DEFINITE INTEGRALS CONTAINING A PARAMETER.

BY PROFESSOR D. C. GILLESPIE.

(Read before the American Mathematical Society, February 25, 1911.)

A FUNCTION $f(\alpha, x)$ is defined for each pair of values of α and x in the closed region $0 \leqq \alpha \leqq 1$ and $0 \leqq x \leqq 1$. For each value of α in the interval $(0, 1)$ the function $f(\alpha, x)$ is an integrable function of x according to Riemann's definition. A function $F(\alpha)$ is thus defined by the equation

$$F(\alpha) = \int_0^1 f(\alpha, x)dx.$$

The problem considered in this paper is one of uniform convergence; namely, the determination of the conditions to be imposed on the function $f(\alpha, x)$ in order that corresponding to any positive number ϵ there exist a number δ independent of α such that

$$\text{(I)} \qquad \left| F(\alpha) - \sum_{i=0}^{i=n} f(\alpha, \xi_i)(x_i - x_{i-1}) \right| < \epsilon,$$

$$(x_0 = 0, \quad x_n = 1, \quad x_{i-1} \leqq \xi_i \leqq x_i)$$

for $(x_i - x_{i-1}) < \delta$.

Closely associated with this problem of uniform convergence are, at any rate, two others which lend interest to it. Of these, one is the problem concerning the continuity of $F(\alpha)$. Under the assumption that $f(\alpha, x)$ is a continuous function of α for each value of x, a necessary and sufficient condition that $F(\alpha)$ be a continuous function of α follows from the theory developed. The conditions under which the roots of the equation $F(\alpha) = 0$ are limiting points of the roots of the sequence of equations

$$\sum_{i=0}^{i=n} f(\alpha, \xi_i)(x_i - x_{i-1}) = 0$$

as n becomes infinite is the second problem.

The absence of continuity conditions does not preclude the existence of the inequality (I).

Example (a):

$$f(\alpha, x) = 1 + 2x \text{ for } \alpha \text{ rational},$$

$$f(\alpha, x) = 0 \text{ for } \alpha \text{ irrational},$$

$$F(\alpha) = 2 \text{ for } \alpha \text{ rational}; \; F(\alpha) = 0 \text{ for } \alpha \text{ irrational}.$$

For a fixed value of x, $f(\alpha, x)$ is discontinuous in α at every point of the interval (0, 1), and $F(\alpha)$ is also discontinuous at every point. Inequality (I) nevertheless exists.

On the other hand it is not sufficient for the existence of the inequality (I) that $f(\alpha, x)$ be limited and $F(\alpha)$ be continuous.

Example (b):

$$f(\alpha, x) = \sin\frac{2\pi x}{\alpha} \quad \text{for } \alpha \neq 0, \quad f(\alpha, x) = 0 \text{ for } a = 0,$$

$$F(\alpha) = \int_0^1 \sin\frac{2\pi x}{\alpha}\, dx = \frac{\alpha}{2\pi} - \frac{\alpha}{2\pi}\cos\frac{2\pi}{\alpha} \text{ for } \alpha \neq 0,$$

$$F(\alpha) = 0 \text{ for } \alpha = 0.$$

F(α) is therefore a continuous function of α in the interval (0, 1). Let the law of subdivision be such that the end points of the νth subdivisions coincide with those values of x for which $\sin 2\nu\pi x$ has maximum values, and let $\xi_i = x_i$. There will then exist an integer n greater than any fixed integer m, and a positive number α less than any fixed positive number δ, such that

$$\left| \sum_{i=1}^{i=n} f(\alpha, \xi_i)(x_i - x_{i-1}) - 1 \right| < \epsilon,$$

where ϵ is any preassigned positive number. Since, however,

$$\lim_{\alpha=0} F(\alpha) = 0,$$

the inequality (I) does not exist.

Nor, again, is it sufficient for the existence of the inequality that both $F(\alpha)$ be continuous in α and $f(\alpha, x)$ be a continuous function of α for each value of x.

Example (c):

$$f(\alpha, x) = \frac{\alpha^3}{\alpha^2 + (x-1)^2} \quad \text{for } x \neq 1,$$

$$f(\alpha, x) = 0 \text{ for } x = 1.$$

$f(\alpha, x)$ is then a continuous function of α for each value of x. The function

$$F(\alpha) = \int_0^1 f(\alpha, x)dx = \alpha^{\frac{1}{2}} \operatorname{arctg} \frac{1}{\alpha} \text{ for } \alpha \neq 0,$$

$$= 0 \text{ for } \alpha = 0$$

is continuous in α throughout the interval (0, 1). Let the end points of the νth subdivision fall at

$$0, \quad \frac{1}{\nu}, \quad \frac{2}{\nu}, \quad \frac{3}{\nu}, \quad \frac{4}{\nu}, \quad \cdots, \quad \frac{\nu - 1}{\nu}, \quad 1$$

and let $\xi_i = x_i$ for $i \neq \nu$ and $\xi_\nu = 1 - 1/\nu^2$; then the sum having $F(\alpha)$ for its limits is

$$\frac{1}{\nu} \cdot \frac{\alpha^{\frac{1}{2}}}{\alpha^2 + \left(\frac{1}{\nu} - 1\right)^2} + \frac{1}{\nu} \frac{\alpha^{\frac{1}{2}}}{\alpha^2 + \left(\frac{2}{\nu} - 1\right)^2}$$

$$+ \cdots + \frac{1}{\nu} \cdot \frac{\alpha^{\frac{1}{2}}}{\alpha^2 + \left(1 - \frac{1}{\nu^2} - 1\right)^2}.$$

For $\alpha = 1/\nu^2$ the last term of the sum is $\frac{1}{2}$; the sum of all the terms is greater than $\frac{1}{2}$ since all the terms are positive. The function $F(\alpha)$ is continuous and equals zero when α equals zero, the inequality (I) therefore does not exist.

THEOREM I. *If $f(\alpha, x)$ be a continuous function of the two variables α and x, the inequality* (I) *exists.*

This theorem is included in the following more general

THEOREM II. *If $f(\alpha, x)$ be continuous in α uniformly with respect to x, i. e., if corresponding to any positive number ϵ there exists a δ independent of x such that $|f(\alpha + h, x) - f(\alpha, x)| < \epsilon$ for $|h| < \delta$, the inequality* (I) *exists.*

This theorem again is a special case of theorem (III).

Theorems I and II are stated under the assumption that $f(\alpha, x)$ is an integrable function of x for each value of α. In Theorem III we drop this requirement and assume only that $f(\alpha, x)$ is a limited function of x for each α.

THEOREM III. *If $f(\alpha, x)$ be continuous in α uniformly with respect to x, then the upper sum* converges to the upper integral* uniformly with respect to α.*†

Since $f(\alpha, x)$ is a limited function of x for each α, the upper integral exists and a function $\bar{F}(\alpha)$ is defined by the equation

$$\bar{F}(\alpha) = \int_0^{\bar{1}} f(\alpha, x)dx.$$

The upper limit of $f(\alpha, x)$ for values of x in the interval (x_i, x_{i-1}) is denoted by $\bar{f}_i(\alpha)$. Unless the upper sum converges to the upper integral uniformly in α throughout the interval $(0, 1)$, there exists a point $\alpha = b$ in this interval such that in any arbitrarily small interval about b the convergence is not uniform. It is sufficient then to establish uniform convergence in the neighborhood of b.

$$\left| \bar{F}(b+\eta) - \sum_{i=1}^{i=n} \bar{f}_i(b+\eta)(x_i - x_{i-1}) \right|$$

$$\leqq \left| \sum_{i=1}^{i=n} \bar{f}_i(b)(x_i - x_{i-1}) - \bar{f}_i(b+\eta)(x_i - x_{i-1}) \right|$$

$$+ | \bar{F}(b) - \Sigma \bar{f}_i(b)(x_i - x_{i-1}) | + | \bar{F}(b+\eta) - \bar{F}(b)|.$$

Corresponding to any positive number ϵ, there exists a number δ_1 independent of i such that

$$| \bar{f}_i(b) - \bar{f}_i(b+\eta) | < \frac{\epsilon}{3}$$

for $| \eta | < \delta_1$. This follows from the assumption that the function $f(\alpha, x)$ is continuous in α uniformly with respect to x. The existence of the upper integral

$$\int_0^{\bar{1}} f(b, x)dx$$

renders it possible to choose a number δ_2 such that

$$\left| \bar{F}(b) - \sum_{i=1}^{i=n} \bar{f}(b)(x_i - x_{i-1}) \right| < \frac{\epsilon}{3} \quad \text{for } (x_i - x_{i-1}) < \delta_2$$

* Cp. Pierpont's Functions of a Real Variable, vol. 1, p. 337; Hobson's Functions of a Real Variable, p. 339.

† The same theorem holds, of course, for the lower sum and lower integral.

$$|\bar{F}(b+\eta) - \bar{F}(b)| = \left| \int_0^{\bar{1}} f(b+\eta, x)dx - \int_0^{\bar{1}} f(b, x)dx \right| \leqq \left| \int_0^{\bar{1}} \{f(b+\eta, x) - f(b, x)\} \right| dx$$

$$\leqq \text{the upper limit of } |f(b+\eta, x) - f(b, x)|.$$

Hence there exists a number δ_3 such that

$$|\bar{F}(b+\eta) - \bar{F}(b)| < \frac{\epsilon}{3} \quad \text{for} \quad |\eta| < \delta_3.$$

Thus we have established the existence of a number δ such that

$$\left| F(b+\eta) - \sum_{i=1}^{i=n} \bar{f}_i(b+\eta)(x_i - x_{i-1}) \right| < \epsilon \quad \text{for} \quad |\eta| < \delta$$

and $(x_i - x_{i-1}) < \delta$.

THEOREM IV. *If the function $F(\alpha)$ be continuous in the interval $(0, 1)$, and the function $f(\alpha, x)$ be a continuous function of α for each x, then under any fixed law of subdivision there will correspond to any integer m and any positive number ϵ an integer $n > m$ and a number δ such that*

$$\left| F(b+\eta) - \sum_{i=1}^{i=n} f(b+\eta, x_i)(x_i - x_{i-1})^* \right| < \epsilon \text{ for } |\eta| < \delta.$$

THEOREM V. *If $f(\alpha, x)$ be a continuous function of α for each x, $F(\alpha)$ will be continuous at b provided that, corresponding to any positive number ϵ and any integer m, there exist a number δ and an integer $n > m$ such that*

$$|F(b+\eta) - \sum_{i=1}^{i=n} f(b+\eta, x_i)(x_i - x_{i-1})^*| < \epsilon \text{ for } |\eta| < \delta.$$

The proofs of Theorems IV and V are almost identical with the proofs of the two theorems which establish the necessary and sufficient condition that the sum of an infinite series of continuous functions shall be a continuous function. These theorems in infinite series, as well as those stated here for definite integrals containing a parameter, are applications of

* In the function $f(b+\eta, x_i)$ x_i may be replaced by ξ_i provided the manner of assigning ξ_i is prescribed.

the following theorem concerning functions defined by sequences of continuous functions. We assume that each of the sequence of functions $\varphi_n(x)$ is continuous in the interval (0, 1), and that $\lim_{n=\infty} \varphi_n(x) = \varphi(x)$ exists. A necessary and sufficient condition for the continuity of $\varphi(x)$ in the interval (0, 1) is that, corresponding to any positive number ϵ and any integer m, the condition $| \varphi(x) - \varphi_n(x) | < \epsilon$ is satisfied for every value of x in (0, 1), where n has one of a finite number of values all greater than m, the value to be given to n depending on the value assigned to x.

CORNELL UNIVERSITY,
March 2, 1912.

ON THE V_3^3 WITH FIVE NODES OF THE SECOND SPECIES IN S_4.

BY DR. S. LEFSCHETZ.

(Read before the American Mathematical Society, April 27, 1912.)

CUBIC varieties in four-space were first investigated by Segre, in two memoirs* which are still classic, and in which he gave a generation of those having more than six nodes, especially the one with ten nodes, while he also considered varieties containing a plane, and gave some of their properties. Castelnuovo† investigated also the V_3^3 with ten nodes, and a good account of the theory of the latter is to be found in Bertini.‡ So far as we know however, varieties having nodes for which the hypercone tangent degenerates into one cut by any V_3^1 in a cone—points which we define as nodes of the second species—have been but little considered. In a previous paper § the writer has given the maximum of these nodes for surfaces, or rather a method for obtaining it. This method admitted of an evident extension to n-space, and in particular gives for V_3^3 in four-space, a maximum of these nodes equal to half the number of absolute invariants of the most

* "Sulle varietà cubiche," *Memorie dell' Academia di Torino*, ser. 2, vol. 39 (1888). "Sulla varietà cubicha con 10 punti doppi," *Atti di Torino*, vol. 22 (1887).

† "Sulle congruenze dell 3° ordine," *Atti dell' Ist. Veneto*, ser 6, vol. 6 (1888).

‡ Geometria proiettiva degli iperspazi, p. 176.

§ "On the existence of loci with given singularities." Read before the Poughkeepsie meeting of the Society, Sept. 12, 1911.

general V_3^3, that is five. We propose, in the present paper, to show briefly the existence of this variety having five such nodes.

The equation of a V_3^3 having 5 nodes, one at each of the vertices of the pentahedron of reference, is

$$\sum_{i,\,k=1}^{i,\,k=5} \frac{a_{ik}}{x_i x_k} = 0 \qquad (a_{ik} = a_{ki}, \quad a_{ii} = 0).$$

The hypercone tangent to the V_3^3 at the point (0, 0, 0, 0, 1) has for its equation

$$\sum_{i,\,k=1}^{i,\,k=4} \frac{a_{ik}}{x_i x_k} = 0,$$

and its section by a V_3^1 will be a cone if we have

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{12} & 0 & a_{23} & a_{24} \\ a_{13} & a_{23} & 0 & a_{34} \\ a_{14} & a_{24} & a_{34} & 0 \end{vmatrix} = 0.$$

If we apply now the transformation

$$X_1 x_1 = X_2 x_2 = X_3 x_3 = X_4 x_4 = X_5 x_5,$$

our V_3^3 becomes

$$\sum_{i,\,k=1}^{i,\,k=5} a_{ik} X_i X_k = 0,$$

that is, a V_3^2, which will pass through the five vertices of the pentahedron of reference, and will be cut by $X_5 = 0$ in a quadricone on account of the vanishing of the above determinant. It will therefore be tangent to $X_5 = 0$. It follows that to show the existence of a V_3^3 with five nodes of the second species, we need only construct a V_3^2 both inscribed and circumscribed to a pentahedron formed by five V_3^1. For this purpose let us consider an arbitrary V_3^2, and three arbitrary V_3^1 tangent to it, which we will denote by A, B, C. We want to find two more V_3^1, say D and E, such that together with the three given ones they satisfy the required condition with respect to V_3^2. Let L be the right line intersection of A, B, C; P_a, P_b, P_c the planes (BC), (CA), (AB); and C_a, C_b, C_c the conic intersections of these planes with V_3^2. These three conics have in

common the points α, β where L cuts V_3^2. With the conditions imposed it is clear that D must go through α (or β) and E through β (or α). . . . Suppose then that D goes through α and E through β. D and E are to be determined by the conditions that they shall be tangent to V_3^2 and such that the plane (DE) shall meet C_a, C_b, C_c respectively in one point. Consider now a fixed point a on C_a, and a point b on C_b. There are two V_3^1 going through a, b, α and tangent to V_3^2, and if c is the point other than α where one of them meets C_c, there are two V_3^1 tangent to V_3^2 and going through a, c, β. If b' is the point where one of them cuts C_b, it is seen at once that (b, b') are in (4, 4) correspondence, and for any of the 8 coincidences it is evident that we have two hyperplanes D, E which together with (A, B, C) form a system of the kind required.

It may be remarked in passing that

$$\prod_{i=1}^{i=4} x_i \sum_{k=1}^{k=4} \frac{a_k}{x_k} + x_5^3 = 0$$

represents a V_3^3 with four nodes of the second species, and is a mere generalization of the cubic surface with three such nodes represented by

$$x_1 x_2 x_3 + x_4^3 = 0.$$

We reserve for a later occasion the consideration of the special cases that may arise in the construction given above.

LINCOLN, NEB.,
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WHAT IS MATHEMATICS?

Principia Mathematica. By ALFRED NORTH WHITEHEAD and BERTRAND RUSSELL. Vol. I. Cambridge, 1910. Royal 8vo. xvi + 666 pp. $8.00.

THE game of chess has always fascinated mathematicians, and there is reason to suppose that the possession of great powers of playing that game is in many features very much like the possession of great mathematical ability. There are the different pieces to learn, the pawns, the knights, the bishops, the castles, and the queen and king. The board possesses certain possible combinations of squares, as in rows,

diagonals, etc. The pieces are subject to certain rules by which their motions are governed, and there are other rules governing the players. A treatise on chess contains all these. Further however it also contains openings which have been found to be advantageous to one or the other of the players and usually contains also various endings of games for the tyro to analyse, in order that he may see how to acquire skill in foreseeing the situations that may arise in any game. One has only to increase the number of pieces, to enlarge the field of the board, and to produce new rules which are to govern either the pieces or the player, to have a pretty good idea of what mathematics consists.

In mathematics the game is much more complicated. The pieces we handle are the members of ranges of a more or less elaborate character. These members may be numbers, functions, lines, operations, any set of things we can define.* The moves on the board are groups of operations that may be performed upon these ranges and their members. We also must take into account a feature which is present in the game of chess in one move only—that of castling. In mathematics we may handle whole combinations of elements and operations upon them, as if they were single things. That is, we must take into account complexes of operations and ranges.

With these elements we do different things, according to our taste and ability. First of all there are the developments of *structure*. These include the construction of magic squares, and other questions of tactic, arrangements and combinatory analysis, factoring, decomposition of fractions, congruences, residues, and theory of form, through the structure of groups, up to finite fields, multiple algebra, calculus of operations, symbolic logic and general algebra. Then there are developments of the *invariants* that occur in different structures. We study algebraic and arithmetic forms, group characters, projective geometry, differential forms, topology, geometry in general, and operational invariants. (We mention necessarily only a few sample cases of the problems referred to.) There is also the study of correspondences of various types, the whole field of analysis or study of *functions*. In this are such things as the functions of a real variable, trigonometric series, algebraic functions, general analysis, geometrical transformations,

* Cf. E. H. Moore, "Introduction to a form of General Analysis," New Haven Math. Colloquium, 1910, page 2.

automorphic functions, calculus of variations, functions of a complex variable, vector fields, differential geometry, and rational mechanics. Further, and most difficult, are the studies in *inversion*. It is here that the expansions of the mathematical game take place. In this line of investigation we find algebraic corpora, ideals, modular systems, differential equations, integral equations, functional equations, and inverses of all kinds. Any theory of mathematics that would be complete is forced to account for all these different studies. To consider, for example, that one has laid the foundations of mathematics when he has produced the irrational number, is to confuse the theory of real variables with mathematics. Important as the problems of the continuum may be, the continuum is not the basis or foundation of structure. A knowledge of the different grades of ensembles does not enable one to ascertain whether a group is compound or not, or whether a number is prime or not. It does not determine the list of invariants of the decic, nor does it develop the differential parameters of a differential form of the fifth degree. The continuum has little to do with the properties of automorphic functions as functions. Cardinal and ordinal ranges do not play a prominent part in modular systems, nor in algebraic ideals, nor in functional equations. Neither likewise does a set of postulates for geometry, or some type of geometry, assist in determining how many associative algebras there are, built on twenty-four units. Indeed the postulates for associative algebras in general do not do this. So it becomes evident that when one wishes to discuss the principles of mathematics he must state what it is he refers to. If the analysis above (which was only indicated in a broad way) is correct, he may discuss the *ranges* with which mathematics has to deal, or he may discuss *operations* in general, or the principles may be those at the base of *multiple algebra*. He may mean the principles of mathematical composition and *form*, or the principles of the *invariance* in the transformations of forms, or the principles of *functionality* and correspondence in general, or the principles upon which may be founded the theory of *inversions*. For example, Russell's Principles of Mathematics was unable to handle the problem of the introduction of the imaginary into mathematics, and endeavored to crowd the whole theory of hypercomplex numbers into the theory of dimensionality. Peano's Formulaire did better,

for the imaginary is defined at least by one of its examples. But the imaginary and the quaternion, and all other associative hypercomplex numbers, not to speak of those not associative, receive scant recognition in either case. A Principia Mathematica should cover the field, or it ceases to justify its title.*

Further we must not confuse mathematics and mathematical reasoning. It is true we infer in mathematics. But we also infer in physics, and history, and in daily life. Mathematics has no copyright on the process. To define mathematics as the science that draws necessary conclusions,† or as the class of all formal implications,‡ does not define at all. Other branches of human learning draw necessary conclusions, and formal implication is not unknown to them. Mathematics also uses the constructive imagination, the generalizing power, the intuition, and other mental processes. To define mathematics as "the study of ideal constructions (often applicable to real objects) and the discovery thereby of relations between the parts of these constructions, before unknown," is better.§ This definition brings out the *ideal*, the *construction*, and the *discovery*, three features which are essential to mathematical development. For example, an abstract group is ideal. Functions of the roots of an equation are ideal constructions. The discovery lies in seeing that the properties of the one will explain the relations involved in the other. Again, algebraic numbers are ideal. We construct with them domains of rationality, and arrive at the Galois theory in the relations.

However attention should be drawn to the fact that among the ideal elements with which we deal are many we invent or create entirely new. Examples are easily found. Quaternions were the result of Hamilton's attempt to extend the number field. The non-euclidean geometries were the attempt to create a new geometry. If any one fact stands out prominently in mathematical investigation it is this fact of the creation of new realms of investigation. Whether these ever are applied to real objects is a matter of less importance

* Cf. Poincaré, *Rev. Mét. et Morale*, vol. 13 (1905), pp. 815–835; vol. 14 (1906), pp. 17–34, 294–317, 866–868. Hilbert, Verh. III. Int. Math. Kongr. Heidelberg (1904), pp. 174–185. Russell, *Rev. Mét. et Morale*, vol. 14 (1906), pp. 627–650. Bôcher, BULLETIN, vol. 11 (1904), pp. 115–135.

† B. Peirce, *Amer. Jour. Math.*, vol. 4 (1881), p. 97.

‡ B. Russell, Principles of Mathematics, p. 3.

§ Century Dictionary (C. S. Peirce).

mathematically. It is a simple affair to invent even a new logic and mode of inference. Thus, let us imagine that the contradictive process were of period three, in place of two. That is, to the proposition p there is a first contradictory p', whose first contradictory is p'', the contradictory of the last being again p. What becomes of the conventional logic now? Yet by symbolism we can develop this kind of logic as well as any. This process we may call the mathematicising of logic. Indeed the volume before us really does something of this kind for logic, as the doctrine of types is close to the example above, and the result is labeled mathematical logic.

The outcome is interesting to mathematicians for several reasons. First of all it is a very general or abstract branch of mathematics. Secondly this book whose first volume is under consideration takes the place of a second volume of the Principles, in which the attempt was made to reduce all mathematics to symbolic logic, or as it is now called logistic. It is expected to demonstrate formally from these notions the derivation of the properties of cardinal and ordinal integers, irrationals, series, and eventually geometry and dynamics. We desire to examine the book from a purely mathematical point of view as to the success of the attempt.

There is an Introduction of three chapters. In some cases the fuller development farther on must be read in order to see exactly what the explanations of the Introduction mean. The first chapter gives a preliminary explanation of the notations used. These symbols are able to be themselves the elements of the entire development, and are thus very fundamental. A complete table is given here for the convenience of readers.

$\vdash$ it is true that..	$\dot{-}$ negative of relation ()
* chapter	$\subset\cdot$ relation () contained in.
$\sim$ contradictory of..	$\mid$ relative product of..
v ..or..	R' the referent as to relation () of
. ..and..	$R''\alpha$ the referents of class α
$\supset$ ..implies that..	$\breve{R}$ converse relation of R
$\supset_x$ ..implies that for x, formal implication	$\exists!$ there is a case of relation
	$\dot{\Lambda}$ null-relation
$=$ ..is identical with..	$\dot{V}$ universal relation
$=$. *Df* nominal definition	$\upharpoonright$ relation with domain limited to a class
$\equiv$ .. is equivalent to..	
$\equiv_x$ .. is equivalent to.. for x, formal equivalence	$\upharpoonleft$ relation with converse domain limited to a class
P_Δ P-selections	$\upharpoonright\upharpoonleft$ relation with field limited to a class
$1 \rightarrow 1$ one-one correspondence	

$1 \rightarrow$ Cls one-many correspondence
Cls name for classes
Rel name for relations
Cnv converse
Ex existent
Cls^2 class of classes
D' domain of relation..
$\breve{\text{C}}$ converse domain of..
C' field of..
$\rightarrow$, sg, referents of..
$\leftarrow$, gs, relata of..
$x \mathbin{\wr} $ operator of x on ()
$\mathbin{\wr} y$ operator of () into y
$\alpha \mathbin{\wr} y$ values of $x \mathbin{\wr} y$ for x over range α
Rl subrelation
Rl ex existent subrelation
φa proposition about a
φx proposition about a variable
$\varphi \hat{x}$ propositional function
$\varphi ! \hat{x}$ predicative function
$(x) \cdot \varphi x$ proposition is true for all individuals x
$(\exists x) \cdot \varphi x$ proposition is true for some individuals x
$(\iota x) \varphi x$ the x with property φ
$\hat{x}(\varphi x)$ class defined by φ
Type range of x such that φx is significant.
$\dot{\cap}$ relation () and relation ()
$\dot{\cup}$ relation () or relation ()
$\downarrow$ couplet relation, vid.
$\cap$ common subclass
$\cup$ common superclass
$-$ negative of a class
$\subset$ contained in...
V universal class
Λ null class
$\exists !$ there is a member of..
E! the member of .. exists
1 the class of unit classes
$1(\alpha)$ the class of unit classes of type α
φ, ψ, χ, θ, functional signs
ι the...
ϵ is a member of the class determined by..
ι class of one member, unit class.
$\iota' \alpha$ the only member of α
Greek capitals, constants
Small Greek letters, usually classes
Capital italics, variable relations
p, q, r propositions
f, g functions
$t' x$ type in which x is contained
$t_0' \alpha$ type in which α is contained
$\text{p}' \kappa$ product of classes
$\text{s}' \kappa$ sum of classes
Cl subclass
Cl ex existent subclass
$'$ of

These symbols may be viewed in two different ways. They may be looked upon as furnishing a system of short-hand, or pasigraphy and stenography combined, intelligible to the initiated, and not only abbreviating the writing, but furnishing a mode of expression in which the usual color, shading of meaning, and associations of words are missing. This in itself would justify their use. Or we may look upon them as being symbols for the abstract elements of reasoning which have been found by the analysis, and for which no appropriate name exists. The latter formal view would not be taken by those who dislike to think that mathematics is the theory of certain combinations of symbols. But it seems to us that if we take the formal point of view, we are doing no more than when we define a rational fraction as a couple of numbers, subject to certain rules, and are able then to identify integers with those couples whose second number is 1. The foundations of arithmetic become solid. So too here if we consider that these symbols themselves, as representative of certain well-defined terms, are under consideration, we shall find a gain in clearness. In fact obscurity arises easily if we try to interpret some of the statements of the book in other ways.

We will undertake to give some notion of the ground covered in this first volume. All we can do of course in our limits of space is to discuss some of the prominent features of the book. The first thing we must consider is the meaning attached to certain words that are not used in the usual sense. The one we encounter at the beginning is the term *implication*. To this we need to devote a careful study, for it is the real basis of the further development.

Implication.

The startling statement is made early that "Newton was a man" and "The sun is hot" are equivalent propositions. From the definition of implication we see also that "Newton was not a man" implies that "The sun is hot," and "The sun is cold" implies "Newton was a man," or $2 + 2 = 4$, or "John Smith killed Pocahontas." This is a very different thing from what most of us would naturally call equivalence or implication. Implication, as used here, is a relation between two elementary propositions, or statements about particular individuals. We might raise the question as to whether there are any such propositions after all. But accepting for the time the assertion that they exist, implication is merely the statement that either the first proposition is false or else the second one is true. Now in the highly special sense in which we find terms used throughout the book, we feel instinctively that there is a certain artificial quality about every definition given or term used. Thus while we find the terms *true, false,* and *not true,* used, as well as *truth-value,* we find no real explanation given of the meaning of these words. Indeed we find later that there is a varying truth dependent upon the order of the statement. We must conclude then that the text throughout is concerned with a certain quality of the propositions considered, and not with the propositions themselves. When the authors talk about p and q they do not mean to discuss the significance of p and q but only a certain quality of p or of q. If the argument is about "Newton was a man," any other true proposition would do as well, for example, $2 + 2 = 4$. The assertions are not about the content of the propositions in either case but about the quality attached to either called its "truth-value." All propositions with the same truth-value are equivalent. They may in any implication be substituted, one for another. This ex-

plains how it is that "Newton was a man" and "The sun is hot" are equivalent. It also shows that the first implies the second, for either the first, irrespective of its significance, has the truth-value falsehood, or else the second has the truth-value truth. It is explained later (page 120) that we may throughout substitute the number 1 for any proposition with the truth-value truth, and the number 0 for any proposition with the truth-value falsehood, and reduce all formulas to the arithmetic of 0 and 1. The notion of truth-value is due to Frege,* although something similar is to be found in Boole.†

With regard to this view, we might suggest that whether the symbols p and q are to be regarded as equivalent or not depends upon the relationship they possess to other things in the universe, as well as upon their own significance. It would seem better to have used a different term to designate what the authors have in mind. A specific symbolism will make the matter clear. Let us agree to mark every proposition either with ° or with ′. This property of being tagged we will call T, thus

T (Newton was a man) is ′, T (The sun is cold) is °

We may then state that what is meant by implication is one of the alternatives

$$\mathrm{T}(p) \text{ is } ° \text{ and } \mathrm{T}(q) \text{ is } °, \text{ or } \mathrm{T}(p) \text{ is } ° \text{ and } \mathrm{T}(q) \text{ is } ′,$$

$$\text{or } \mathrm{T}(p) \text{ is } ″ \text{ and } \mathrm{T}(q) \text{ is } ″$$

It is to be observed that $\mathrm{T}(q)$ is tagged ′ if $\mathrm{T}(p)$ is ′, otherwise it may be either. The one case excluded is evidently: tag of p is ′ and tag of q is °. We may admit that this is a simple thing, but (while it may be an idiosyncrasy on our part) it does not seem to be elementary. The notion of two tags, and of the property of being tagged, are clearly involved, and unless we make the tagging a purely haphazard affair, there is also involved the problem of determining *which tag must be placed on a given symbol.* The basal assumption (not mentioned) of the entire book seems to be, that we are in a position to say with regard to any proposition p about some specific thing whether it is true or not. Often it is the truth that we are

* Russell, "The theory of implication," *Amer. Jour. Math.*, vol. 28 (1906) p. 160.

† Laws of Thought, 1854, p. 70.

endeavoring to discover. In fact, we have here the first example of the assertion that some parts of the book become more clear if we treat the results as purely symbolic, the whole being a calculus of symbols. If we know how to tag q there is no use in mentioning p. If p happens to be false, or if p happens to be true, we still confront the fact that we must decide that we have to choose between exactly the alternatives stated above. Hence for practical inference, this kind of implication seems to us to be worthless, and we therefore think another name would be desirable. The definition of implication, or rather the meaning given the implication sign, by Peano,* seems to be more fundamental, and elementary. It reads thus:

$$p)q$$

de p on déduit q; si p, alors q; la p a pour conséquence la q; la q est une conséquence de la p; la p est une condition suffisante de la q; la q est une condition nécessaire de la p.

We deal here directly with the propositions p and q and not with any functions of qualities they may possess. From the formal point of view, however, we have obtained a two-valued function of the indefinite set of marks, p, q, r, etc., that is, T( ) is $'$ or $^\circ$.

The propositional form that has a variable argument is generally expressed by φx, and we come next to the implications corresponding, that is, to *formal implication*. According to the Principles of Mathematics, the formal implication is the main thing in mathematics. It means that in φx we substitute for x any symbol (later it is restricted to a given type). The propositions resulting will each have a tag, as $'$ or $^\circ$, that is, as true or false.† The same is done with ψx, where ψ is a form into which we put the variable x in order to arrive at a proposition. Each of these propositions is tagged. Then φx implies (formally) ψx if in each case the tag on φx is $^\circ$ or else the tag on ψx is $'$, the same x occurring in each of the two. In other words, we must be able to assign for any given x the proper tag for φx and for ψx, and if we make out a three column table in which the first column is marked x, the second φx, the third ψx, and then enter the values of x as arguments, and the tags as values

* Formulaire, 4 éd., page 4; 5 éd., p. 3.

† If absurd, they are not propositions.

in the proper columns, we will necessarily have in the second column both ° and ′, and in the third column also both ° and ′, but with ′ in the second will always come ′ in the third. We might also state it thus: ψx must be true at least whenever φx is true. It is explained that we do not need to know or to produce every x about which the proposition may be stated. The φ and ψ are taken intensively. This is the first case we note in which there seems to be a lack of agreement between the theory and the practice. The definition calls for a comparison of the tags on two sets of expressions, which from their character would usually be infinite in number. As the direct comparison is impossible, the practical application goes back to a problem in intension, a term the authors endeavor to the utmost to shut out of the book. Again, if φx happens to be false for x in every case, we nevertheless have ψx implied. This may be true, that from false propositions anything may be concluded, but it does not advance mathematics very much. It would seem therefore that the attempt to found the whole system on the principle of truth-values (which we have called tags) is not so very successful, and that it would be better to make the undefined implication the base of the system. Indeed the authors apparently fall into this habit unconsciously. Thus we find as one of the assertions of the book

$$*2.04 \vdash :.p.).q)r:):q.).p)r,$$

which they interpret: if r follows from q provided p is true, then r follows from p provided q is true. This reversal of conditions in the theory of functions would work havoc only too frequently. Of course the reading should be: consider {the tag on p is ° or else the tag on the statement [either the tag on q is ° or that on r is ′] is ′}, then if we mark all that has just been stated in { } with ′, we must mark also with ′ all that follows, viz. {the tag on q is ° or else the tag on the statement [either the tag on p is ° or else the tag on r is ′] is ′}. This is quite different from the Peano reading given just above.

We have dwelt upon the idea of implication as set forth here because this idea seems to be used more as a test of the accuracy of the results obtained than as a working notion. It is held in reserve as a court of last appeal. If one starts in directly with Section A and not with the Introduction, he does not encounter the notion of truth-value until *4.01 on page 120. If implication is taken as the fundamental notion

and left undefined, we can define all the other symbols in terms of it and contradiction. This was done in the Principles, although the method used seems unnecessarily cumbrous. However, truth-value does not appear in the symbolism, and we have practically gained the following fundamental symbols:

φa, φb, etc., definite propositions about constant subjects a, b, etc.

φx, ψx, etc., definite propositional functions but variable arguments.

$)\cdots$, a relation between propositions, called implication.

Propositional Functions.

The propositional function is very important. It not only includes the usual predicate but may be any kind of a form with a blank place left for the entry of the argument. It is a symbol for the process that enables one to pass from a given argument term to another, the value term. The notation is as follows:

φa, a has the property φ, or of a we may say φ. This sentence is the value of φ for a.

φx, the propositional function applied to a variable argument. This is a symbol for any *one* of the values of the function, including statements which are not true as well as true statements.

$\varphi\hat{x}$, the function itself, as function. The $\hat{x}$ appears merely to assure the reader that the function really is a function of something or other. If the authors could have brought themselves to accept the Frege* notion and symbol $\varphi()$, the apparent argument could have been omitted.

$(x)\cdot\varphi x$, the entire list of values of φx are represented by this sign. In a large majority of them the truth-value would be ° of course. Also the x is restricted to the range called the type of φ. The expression reads "φx is every case where x belongs to the type of φ."

$(\exists x)\cdot\varphi x$, there are values of x which give φx the truth-value $'$.

$\hat{x}(\varphi x)$, this symbol seems to have two meanings, at war with each other. In the early part of the book it is defined to mean the class (aggregate, ensemble) which consists of those arguments that make φx true. In other words, to be the set of

* See Principles, page 505.

individuals in the class determined by, or defined by, the function φ. Later it is identified with a symbol $\psi\hat{x}$ which is purely a function symbol, and does not represent individuals at all. This symbol practically defines the class property. The latter meaning seems to be the one which the authors expect to use, and may be interpreted to be the class as class, and not as individuals, but considered as a *denoting* symbol. This use of the symbol to represent the predicative function that would define the class collectively seems to be necessary in the system to enable us to use classes as arguments of functions. It is explained that we do not arrive at real classes thus, but only incomplete symbols. For example it is something like this. If we desire to say "The governors of states all wore silk hats," we must recast the statement to read "Certain persons were silk-hatted governors of states." This use of the class symbol $\hat{x}(\varphi x)$ and the function symbol $\varphi\hat{x}$ is close to that of the phrase "governor of a state," in two different senses, one meaning defining the qualifications necessary to be the governor of each of the states, the other defining the actual governors, so that they could be identified among other men. Both are functions. On this basis there are *no classes*, although the word class appears everywhere in the book. However, the claim is made that we have something just as good as a class, and in fact (page 84) "in mathematical reasoning, we can dismiss the whole apparatus of functions and think only of classes as "quasi-things," capable of immediate representation by a single name." An example would be the imaginary points of a curve.

However that may be as a matter of interpretation, we at least have arrived at two more symbols, from a mathematical point of view: the proposition as function, and the definition of solutions of a proposition. We have, in brief, isolated the function sign φ, and we can speak of "x such that φx." It would seem now that the fundamental thing after one has exhibited his set of elements with which he proposes to work, would be to consider functions of one variable, then synthetic processes by which these may be built up into useful structures. The importance of the propositional function is sufficiently insisted upon, but the uselessness of a mere set of isolated individuals is not dwelt upon. Nor is the character of synthetic processes in general examined at all. Of this we shall have something further to say.

Extensions.

Considerable stress is laid on the assertion that mathematics is primarily concerned with extensions, despite the fact that classes have become mere ghosts of themselves. An example would be any geometric theorem from Euclid. As "Let triangle *ABC* be isosceles, then triangle *ABC* has equal base angles." We are apparently discussing the single triangle *ABC*, but in reality we expect what we say to hold good for any triangle and thus for every triangle. We have the formal implication, which holds between the two sets of elementary propositions, one for each and every triangle. The ground of the reasoning seems to be an ambiguous case, and we seem to reason from any one to all. But is it so? If in a complex mental structure one chooses to pay attention only to certain features of the structure, and discuss them, ignoring the other accompanying features, are the statements about the whole complex structure, or about the portion abstracted? In proving the theorem cited above, does the color of the crayon used in drawing the triangle also enter the argument? Does the size of each individual angle and side also enter, or only certain relations they have? In finding the limit of the expression $2-(\frac{1}{2})^n$ do the particular values that n may take enter the argument at all? If one must answer no in these cases, that is to say, if one can abstract at all and reason about his abstraction, then we see no force in the constant appeal to extension. Is not the propositional function of the nature of an invariant rather? We say: *this* triangle has two equal sides, so *that* triangle, so also *yonder* triangle. In all the propositions of this sort that we choose to write down, we find the invariant phrase: two equal sides. It does not appear to be essentially different from any other invariant. If we were to conceive a transformation that could convert *this* into *that*, and *this* into *yonder*, the invariant of the transformation would in this case be: two equal sides, that is, isoscelism. Now the real question is, whether we can discuss invariants apart from the other circumstances in the concrete cases in which they are invariant. We certainly do this in mathematics. Indeed, is not this present book an attempt to discover what are the logical invariants in mathematics? In mathematics we are dealing with abstract elements all the time. The world of mathematics is an ideal, that is, an abstract world. We either abstract it from what

we find in experience, or we create it de novo. For example, no such thing as *two* exists outside a mathematical mind, any more than the rainbow exists as color out there in the sky. We build our own structures and determine their relations, and no abstraction can be said to be more abstract than any other. In Socrates is mortal, Aristotle is mortal, Charlemagne is mortal, Socrates, Aristotle, and Charlemagne are abstract. No one would call them real men. They are as abstract as mortal, and as abstract as all men. Socrates is a word which represents the persistence of certain qualities of something day after day, and is thus in itself an invariant and therefore an abstraction. In one sense the whole of mathematics is the study of invariants.

If this analysis is correct, then when we say that the property φ implies the property ψ we do not need an extension to which to refer it. It is immaterial whether the extension is there or not. We can study the property as well in one case, as in a million, or an infinity of cases, if only we can isolate it itself. That is what we do in geometry, and in fact all through mathematical thinking. From this point of view (which may coincide with that which the authors denominate the philosophical, and with regard to which they admit the contention) mathematics is more concerned with intensions than extensions. These remarks apply to the extensional functions of functions. Indeed the paragraph in the middle of page 77 says:

". . . the functions of functions with which mathematics is specially concerned are extensional and . . . intensional functions of functions only occur where non-mathematical ideas are introduced, such as what somebody believes or affirms, or the emotions aroused by some fact."

It is difficult to see in this any more than the assertion that mathematics is not concerned with non-mathematical ideas, which no one would pretend to deny, unless it be some philosophers who thought that the Principles of Mathematics did not discuss mathematical ideas.

But there is a further point that we must notice. It is that the definition of function and class does not really produce the members of the extension at all. If we speak of the points of intersection of $x^2 + y^2 = 25$ and $x^2 + y^2 = 36$ what points are given as the extension of this proposition? If we speak of the roots of the equation

$$x^6 + 2x^5 - 13x^4 + x^3 - 7x^2 + 11x + 17 = 0,$$

where is the process in all the development of symbolic logic that will determine them for us? If we ask for the hypercomplex numbers that have the characteristic equation

$$\xi^4 - 4\xi\eta^2 + 5\eta^3 = 2$$

who in all the mathematical world will send us the list? Or lists, we should say, for this one equation will determine more than one certain set of such hypercomplex numbers. Quaternions satisfy the equation

$$q^2 - 2Sq \cdot q + T^2q = 0,$$

but so do other hypercomplex numbers. Take the most definite function we can find, and what does it give? Not a class in extension, but certain properties that are found in some example we may produce, yet which may exist in an infinite number of other examples we have never thought of. Mathematics is full of discoveries of just such expansions of the extension of the notions that we have been using. It is a great advance, as Poincaré says, to find that we can bring two things, that is, two extensions under one name. As a simple case again, the notion of prime number is surely a definite thing, yet who knows how to ascertain whether 67989379012301 is a prime or not? The notion of simple group is definite, are there then simple groups of odd order, other than cyclic groups? When von Staudt called involutions on a line complex numbers, were they cases under the definition or not?

There is, as everyone knows, a vast difference between finding the value of a function for a given argument and finding the arguments that will satisfy a given function. In fact to meet the latter requirement mathematics has had to invent whole new extensions. Symbolic logic does not give us any assistance in this work of development nor any new methods. Its problem is only to criticise the character of the inferences involved in the process of development. It furnishes neither the major nor the minor premise but simply passes upon the validity of the transition from both to the conclusion. And all it furnishes in the propositional function is a sort of common invariant for many sets of classes. The classes may not exist in some senses, and may exist in others, just like the imaginary roots of an equation. But the propositional function does not

point out the particular cases of members of the class it defines any more than it solves an algebraic equation. This is quite different from the explicit mathematical function,* like $\sin x$ or

$$\int_0^\infty \frac{\sin x dx}{x},$$

but analogous to the implicit function, like

$$\int \sqrt{y - x}\, dx, \text{ where } x^3y - xy^2 + y^5 = 4.$$

Types.

Chapter II of the Introduction develops the part of the Principia which differs most from the Principles, the *Doctrine of Types*. By this doctrine the authors hope they have resolved the paradoxes of the Principles as well as others that have been stated in discussions provoked by the original ones or by the theory of ensembles of Cantor.

The net result of the discussion seems to be that the predicate of a sentence is of the nature of a matrix or function symbol, and cannot serve as a subject for a sentence which has it also as predicate. Thus we might consider the statement "Triangle ABC is scalene." The functionality involved here, scalenity, cannot be put as the subject of this sentence. Scalenity is scalene, would be an absurdity. In this form at least, we might all admit that there is a sort of hierarchy of functions, if not of types. Of course one may talk of scalenity, but scalenity does not belong to the range that is itself scalene.

The *type* of a function is the class of objects that make it significant. That is to say in substituting values for x in φx some of them will give true propositions, some will give false propositions, some will give statements that are neither true nor false. That ensemble which produces with φ a proposition, true or false, is the type. The outcome is a little curious, as it leads to a reincarnated ghost of the buried *class*. The last paragraph of page 173 is interesting reading in connection with what has gone before. The difficulty seems to be of the same type as that which certain mathematicians find in recognizing any symbolic operator as an existent entity of the same character as the things it operates upon. Strictly speaking, of course there is a difference between $\frac{5}{1}$ and 5,

* Russell, *Revue Mét. et Morale*, vol. 13 (1905), pp. 906–917.

at least till we come to see that the invariant properties we spoke of above are the same in the one case as in the other, and we identify the two. In just this same way we elevate ensembles in general into the region of functions, and then we can draw a distinction between φ as function or φ as argument.

Whether we agree or disagree with the philosophy underlying the argument, we have gained one point more in our symbolism, that is, that we may make a function sign out of any symbol, simple or complex, and we may use any symbol as an argument for the proper function sign. In other words we may construct more and more complicated forms, and we may substitute for any single symbol or set of symbols, complex symbols. Incidentally we have a relativity theorem in mathematics in the doctrine of types. For no type is the bottom of all types. Types are only relative. An individual in today's discussion may be a function in tomorrow's. But do we resolve a contradiction by calling it absurd?

And it is difficult to see how the doctrine of types can be reconciled with many mathematical developments. Thus if we define the function φ by the differential equation $d\varphi x/dx = \varphi x$, we seem to have a case in opposition to the doctrine. For to define the derivative we must know an infinite set of values of the function φ, which is itself defined by means of the derivative. Apparently then we define a function in terms of the function itself. Another case more to the point possibly is the integral equation, and the integro-differential equation.

Relations and Descriptions.

The third chapter of the Introduction is devoted to descriptions, classes, and relations, under the title Incomplete Symbols. It is said that these can be defined only in their use. They are analogous to the symbols $\int$, ∇, $\sin^{-1}$, etc. However, it is pointed out that "the incomplete symbols are obedient to the same formal rules of identity as symbols which directly represent objects, so long as we consider the equivalence of the resulting variables (or constant) values of the propositional functions and not their identity. This consideration of the *identity* of propositions never enters into our formal reasoning." Under the limitations to the use of these symbols we find that, while x is always identical with x, yet the round square is not

identical with the round square, for the reason there is no round square. This surely resembles the argument: Nothing is better than heaven, a shilling is better than nothing, therefore a shilling is better than heaven.

The descriptive symbol is $(\iota x)\varphi x$. It is used in precisely the same way as, and in fact differs little from, the symbol for a class. That is to say, it is a function sign, denoting the single object in the class. For example, the author of Waverly. This phrase does not point out the individual Scott, yet it identifies him. The legal John Doe does the same thing for a criminal who refuses to give his real name. These are, in the sense defined, function signs.

The relation is also a function, the function however having two arguments. There are many features common to these three incomplete symbols. The logic of relatives has also many developments that are not found in the others.

With regard to the three chapters of the Introduction, we desire to remark that they are in general somewhat difficult to apprehend as they now stand, for two reasons. First the distinctions drawn seem in many cases to be confused. It is sometimes difficult to ascertain whether the authors are using words in an every-day sense, in a philosophical sense, or in a purely technical sense. It is not always clear whether a symbol is under discussion, or the meaning of the symbol, or the use of the symbol. The introduction of many more good examples might have remedied this defect in style. In the second place, the statements do not seem to be thoroughly consistent. For example, it is not easy to decide what the authors mean by extensional. In the early part it seems to mean, as ordinarily in logic, the totality of individuals constituting an ensemble or collection. Later it seems to mean anything intellectual as distinguished from the emotional and the volitional. Then on page 196: "Propositions in which a function φ occurs may depend, for their truth-value, upon the particular function φ, or they may depend only upon the *extension* of φ. In the former case, we will call the proposition concerned an *intensional* function of φ; in the latter case an *extensional* function of φ." Also the apparent dread the authors exhibit towards the word *concept* seems to make the explanations often involved. The root of the whole difficulty seems to lie in an unconscious, or at least unstated, philosophical theory that a general notion is only symbolic

and has no real existence of its own, but is existent only as it is manifested in some supposed concrete form. This notion seems to be the source of many of the peculiar interpretations forced upon the symbols. We may in most cases interpret them otherwise, as we have tried to point out; and as a calculus of logic, the system given here is very complete. We consider next the constructive features of the work.

Elements.

Substantially all that is meant by a *proposition* is to be found in the formal character of the two symbols p and $\breve{}p$. The latter is called the contradictory of p. It is of such a nature that the function $\breve{}$ is involutory, that is, $\breve{}\breve{}p$ is the same as p. If we understand then that we have a set of symbols p, q, r, or φa, φb, φc, and the like, with the duplicate set of their contradictories, we have the elements of the subject as analysed in this book. The descriptive symbol is substantially the same as we find exemplified mathematically in the symbol $^{+}\sqrt{2}$, which in the present notation would be $(\iota x)\ (0<x, x^2=2)$. That is to say, we introduce into arithmetic the indefinite symbol x with the agreement that we will insert it in our number series, and that for x^2 we will always write 2. This is, in a more general case, like the Kronecker modular theory. It is along the line of the algebraicising of mathematics, as opposed to the arithmetising of mathematics. It is immaterial from this point of view whether the thing x exists or not. If in any sense it does exist then we may use a single symbol for it rather than the long form $(\iota x)\ (\varphi x)$. Practically the idea of class is of the same kind. A mathematical example is the definition of a class of algebraic numbers by an equation, as $x^4 + 5x^3 - 2x^2 + 3x - 13 = 0$. The idea of a symbolic class ought not to disturb a geometer, for he is used to imaginary points, circular points at infinity, and the like. We talk of these ideal things as if they existed, being careful that our phrases have meaning when we find the entities do exist in any sense. The same notion occurs again in the relative. A mathematical example is the equation of a curve. Thus $x^2 + y^2 = 25$ furnishes a relation, which finds as its proper representative the *curve* itself, while the pairs of arguments x, y furnish the points on the curve. The curve is a correlating agency for bringing together these pairs of arguments. Whenever there are entities which may be considered to be

represented by these symbols, then we use single letters for them and treat them in the old-fashioned manner of handling classes. This view of mathematics, and of propositions in general, we would prefer to call *functional*, rather then extensional, or intensional. It is surely mathematical, whether it is logical or not, and makes mathematics the fundamental basis of all reasoning, even more than the specialized interpretation called symbolic logic. On the philosophical side it seems to emphasize the statement that all reasoning in the last analysis is not about things but about relations between things, for every function expresses a relation between its argument and its value, and every relation may be referred to an ideal thing at least.

Combinations.

We must consider next the combinations that are actually built up out of these elements. The general development is given in Part I, which extends from page 89 to page 342, and is called Mathematical Logic. We desire to consider it apart from any meaning of a specific character that might be attached to the symbols. There is a single combination introduced, represented thus: $p \vee q$. Any two symbols may be joined in this manner. It is commutative, that is $p \vee q$ is the same as $q \vee p$. Special symbols are used for the combinations $\breve{p} \vee q$, which is written $p)q$, and the combination $\breve{}(\breve{p} \vee \breve{q})$, which is written $p \cdot q$, the first being called implication, the latter logical product, while the basal combination is called disjunction. Any two of these symbols may be omitted. Indeed for ease of manipulation, it seems that to express everything in terms of the symbol $\bullet$ would be best. Thus for $p \vee q$ we write $\breve{}(\breve{p} \cdot \breve{q})$, and for $\breve{p} \vee q$ we write $\breve{}(p \cdot \breve{q})$. We agree further to certain permissible reductions or expansions. Thus, for example, we may write for p,

$$p \cdot p, \quad \text{or} \quad p \vee pq, \quad \text{or} \quad p \cdot q \vee p \cdot \breve{q}, \quad \text{or} \quad p \vee (p \cdot \breve{p}) \cdot q,$$

whatever q may be. In fact if we consider that in any product p and $\breve{p}$ are incompatible and such product may be dropped after the sign v, we arrive at one of the simple methods of handling this calculus. We agree further that if we have any expression φx, where x is variable, we may write the symbol $(\exists x)\varphi x$, and likewise if we have φx and φy(*9.1 and *9.11). Further we agree that in φx we may substitute for x any

symbol a of the same type as x, and also that x may run over a given range of proper type (*9.14 and *9.13). Also we agree that in any expression φa, where a is a constant, we may consider φ by itself as a function, and vice versa (*9.15). We may now take any combination of symbols and build more complicated ones with the use of the v or the •. We add, however, to the symbolism an expression for the range, thus: (x), meaning all values of x, and $(\exists x)$ meaning those values which are solutions. The name formal implication is given to the combination

$$(x)\varphi x)\psi x \text{ or } \varphi x)_x\psi x.$$

In using functions of two variables we introduce the abbreviations called relations. To indicate that x is a solution of φx we abbreviate thus: $x \in \varphi ! x$.

The formulas resulting from these few elements are very numerous, and no attempt will be made to go into them. It is from the results of these combinations however that the authors expect to produce other combinations which will have all the properties of numbers, series, etc.

It is obvious that, out of all the processes the mind goes through, others might have been selected as the fundamental ones. Whether a commutative combination is in the end more useful than one that is not commutative may be a question. We often must use a non-commutative product in mathematics, as "if p is *first* true, and then q is true, it follows r is true." The whole consideration of mathematical form* might from certain points of view be considered to be a prerequisite to the study of any kind of combination. In Whitehead's Universal Algebra this is partly in evidence.

We need to note further that the few modes of combination used here are really supplemented later by a free use of relational symbols, P, Q, R, ϵ, P_Δ, in fact by so many that when we remember the few combinations used here we wonder why these in particular should have been chosen to be represented by arbitrary signs. Indeed the number of arbitrary symbols which have to be memorized is so great in the book that one is willing to conclude that a more significant system could have been worked out. But taken as it is, there remains still the mathematical problem. Stated in brief it is this: given two operations by which from elements or marks new

* Cf. Kempe, *Trans. R. Soc. London*, vol. 177 (1886), p. 1–70; *Nature*, vol. 43 (1890), 156–162; *Proc. Lond. Math. Soc.*, vol. 26 (1894), 5–15.

elements or marks are produced, namely, let us say, given that from p, q we construct

$$s = \varphi p, \quad t = \psi(p, q), \quad \text{and} \quad u = \theta(p, q)$$

with the conditions or identities that

$$\varphi s = \varphi\varphi p = p, \quad \varphi\psi(p, q) = \theta(\varphi p, \varphi q) \quad \text{or} \quad \varphi\theta(p, q) = \psi(\varphi p, \varphi q),$$
$$\psi(p, q) = \psi(q, p), \quad \theta(p, q) = \theta(q, p).$$

We now have the properties of these combinations as combinations to consider. This problem is one of general algebra, universal algebra, or multiple algebra, according to the title preferred, and has at its base the very fundamental question as to what a combination is logically, psychologically, and otherwise; and what the operator φ, or θ, or ψ may be, what it does to the operand, what operators are derivable from it and how; further, what the result of the operation, say s, t, u above, is; how one may pass from the operand to the result, and reversely. These are elements that seem to be overlooked in the development as given in the book. Before one uses a calculus, in other words, he should investigate the laws of his calculus.

In the course of such investigation, it turns out that structural laws are very numerous. We may investigate laws that have been assigned purely arbitrarily, as for example those that actually have been so assigned in the study of multiple algebra. The reduction of all these divers arbitrary types of structure to a few simple forms is not possible, and the introduction of extra relational symbols merely furnishes a symbolism, but does not account for the forms nor does it show that they are deducible from the primitive forms laid down in this book as the basis of all reasoning. And if what is meant is that mathematical form consists of relations, then nothing is done beyond furnishing a mere name to a class of entities. Let us put it otherwise: to single out a few combinations as worthy of special signs, and to represent all others as relations, using letters, does not substantiate the claim that all terms have been defined in terms of the few combinations. The expressions for relations $\hat{x}\hat{y}\varphi(x, y)$ are combinations and the original $p \vee q$, $p \cdot q$, and $p)q$ can be so expressed; for each is not different from $\varphi(a, b)$ for a properly chosen φ, hence are cases of $\varphi(x, y)$, therefore define relations. To say

that ultimately all logic is reducible to propositional functions, would then be the proper outcome. And logic becomes thus a branch of general algebra.

Prolegomena to Cardinal Arithmetic.

This constitutes Part II of the present book. Part III treating of Cardinal Arithmetic, Part IV of Ordinal Arithmetic, Part V of Series, are mentioned for the following volumes. The subjects treated in Part II are of high importance not only for cardinal arithmetic but for the ensemble theory. The divisions are Section A, Unit classes and couples; Section B, Sub-classes, Sub-relations, Relative types; Section C, One-many, One-one, Many-one relations; Section D, Selections; Section E, Inductive relations.

We find 1 defined here by the symbol

$$1 = \hat{\alpha}\,[(\exists x)\cdot\alpha = \iota`x]\ \text{Df.}$$

In words, 1 is the class of all unit classes, or since we have abolished classes per se, we will paraphrase this to read: 1 is a function satisfied by nothing but those functions which are true each in one case only. Or again, one is a property possessed by functions, namely, uniqueness of argument. For example, we speak of the author of Waverly, the President of the United States, the sin 30°, all these enable us to put the word *the* in evidence, and the *the-ness* in their character is that common property called 1. Whether this is a logical definition of the everyday 1 or not, what is accomplished is the construction of a symbol out of those already existing which defines the property of uniqueness. We agree to use 1 in place of the longer form $\hat{\alpha}[(\exists x)\cdot\alpha = \iota`x]$. Likewise 0 is defined to be the function satisfied only by those functions which are never true. For example, () is not identical with itself, () is true when its contradictory is true, and such like. These are in no case true, and the property of their impossibility is the number represented by 0. The cardinal 2 is defined similarly as a function satisfied only by functions which define couples. If the couple defined were an ordered couple, then we define 2_r, the ordinal 2. For example, 2 is a property possessed by quadratic equations, when we confine our attention to the solutions they have. From the purely symbolic point of view we use 0 and 2 to abbreviate the forms

$$0 = \iota`\Lambda \quad 2 = \hat{\alpha}[(\exists x, y)\cdot x \neq y\cdot\alpha = \iota`x \vee \iota`y]\ \text{Df.}$$

Mathematically we have thus found that if the words one, none, two are used, we must have in mind the *uniqueness* of certain classes, the *impossibility* of certain classes, or the *dyad character* of certain classes. We may accept or reject this view of what the symbols mean, but practically we have placed 0, 1, 2, in the list of symbols which form the range of classes of classes. They belong to the second order symbols. The symbolic point of view is nearly the same as saying that we start with objects, these are entities of any order m. Then we make a set of tags to enable us to distinguish the objects without being concerned with their other qualities. These tags are $m + 1$ order symbols (classes). We then make a set of symbols to enable us to talk about the tags. This set of symbols is the set of cardinal numbers, and is of the $m + 2$ order in the process of symbolizing or abstracting.

We begin now to reach arithmetic. As one example of what it looks like, we will quote the theorem which is to prove later that $1 + 1 = 2$. It runs thus

$$*54.43 \vdash : \cdot \alpha, \beta \epsilon 1.) : \alpha \frown \beta = \Lambda \cdot \equiv \cdot \alpha \smile \beta \epsilon 2.$$

That is in English, if α and β are unit-classes with no common members, then their smallest superclass is a couplet.

In Section A we find also the ordinal 2_r, which does not differ from the class of alio-vids of C. S. Peirce; and also the ordinal $\dot{2}$, which does not differ from the class of vids. An ordinal 1 might be defined by $\dot{2} - 2_r$, which is the common subclass of vids and vids that are not alio-vids, that is the class of idem-vids. The connection with the cardinal 2 is in the fact that an alio-vid, that is, an asymmetric relation, must have two distinct terms. The relation here, being a vid, is between one object of thought and one other object of thought. Of course we are not far from the theory of quadrate algebras and matrices in general after we have arrived at this result.*

We pass over the next three sections, although they are of high importance, to Section E, which treats of generalized mathematical induction. The notion of *hereditary* class is defined, μ is a hereditary class with respect to the relation R if successors of μ's are μ's. For example if μ is the peerage, μ is hereditary with respect to the relation of father to eldest son. If μ is numbers greater than 100, μ is hereditary with respect to the relation of ν to $\nu + 1$. Mathematical induction is

* C. S. Peirce, *Amer. Jour. Math.*, vol. 4 (1881), p.221.

evidently included in this class of relations, and by means of it we pass from any finite integral case to any greater integral case, but finite. No such proof holds for any infinity. Thus we may say the binomial theorem is proved in this manner for *any* finite integral exponent, but not for *all* finite integral exponents. Indeed the word *all* here has no sense. We are led to consider powers of relations and the analysis of the field of a relation. This belongs to the general theory of operations.

The whole of the second Part really is mathematical logic of a little more specialized character than Part I, and this first volume could have properly been called a treatise on the mathematics of logic.

Summary.

In summary, the object that we have had in mind was to show that this first volume of the Principia is in reality an application of mathematical methods of definition and synthetic combination to the relationships between the abstract things that logic chooses to discuss. By means of a symbolism, which awkward as it is, is sufficiently comprehensive, a study is made of functions: as related to terms in propositions, and as shown in the particular forms of descriptions, classes, and relations. In one sense the highly ideal character of mathematical objects is made more evident. In another sense the real mathematical object, though already ideal, is sublimated still further into a logical object. The book is a culmination of the critical investigation of mathematical foundations of recent years, and will no doubt advance the systematization and mutual readjustment of mathematical treatments. It will assist in discovering tacit hypotheses, and in putting into formal shape the demonstration of many facts that have been brought to light by the intuition. If it eventually helps in any substantial manner to unify different theories and show their common features it will do enough.*

But while we may admit that it has perhaps placed the fundamental principles of the theory of ranges in a more definite form, and has done something for the theory of relations, we insist that the other great theories of mathematics are barely touched upon if, indeed, at all. These, we pointed

*M. Winter, *Revue Mét. et Morale*, vol. 15 (1907), pp. 186–216. E. Borel, ibid., pp. 273–383. H. Poincaré, ibid., vol. 17 (1909), pp. 451–482, 620–653. H. Dufumier, *Bull. des Sciences Math.*, vol. 35 (1911), pp. 213–221.

out, were the theory of structure and form, the theory of invariance, the theory of functions as functions, the theory of inversions. That these can receive a general treatment we do not doubt, inasmuch as some of them are receiving such development. In logistic then we find only a very definite *branch* of mathematics, and in this volume we have the most complete treatment of logistic that exists. The question that many have asked naturally "How far does it assist in building up synthetic systems of mathematics" is easily answered. It reaches arithmetic only after one volume of 666 pages. We would not expect the complete treatise then to furnish much that would be of a synthetic nature. Indeed that would be as unreasonable as to expect to build Eiffel towers and Eads bridges from a study of postulates and axioms for the foundation of geometry. While design rests upon these things in a sense, design antedates them just as language antedates grammar. It is not fair to the book or its aim to assert that it does nothing synthetic. Its problem is philosophical and analytical. It does enough if it shows us what are the characteristic features of reasoning and generalizes the types of reasoning. In this respect it is scientific as well as philosophical. It examines the rules of the great mathematical game. But it does not play the game nor undertake to teach its strategy.

JAMES BYRNIE SHAW.

DIFFERENTIAL GEOMETRY.

Vorlesungen über Differentialgeometrie. Von LUIGI BIANCHI. Autorisierte Deutsche Übersetzung von MAX LUKAT. Zweite, vermehrte und verbesserte Auflage. Leipzig und Berlin, B. G. Teubner, 1910. xvi + 721 pp.

IN 1899 Guichard announced (*Comptes Rendus* 128, page 232) without proof the following theorems:

I. Let M be a point of a quadric of revolution Q whose axis is of length $2a$, F_1 and F_2 being the foci of Q and φ_1, φ_2 the points symmetric to F_1, F_2 with respect to the tangent plane to Q at M; let S be a surface applicable to Q; as S is applied to Q the points F_1, F_2, φ_1, φ_2 invariably fixed with respect to the corresponding tangent plane to Q take positions which

are denoted by F_1', F_2', φ_1', φ_2'; the loci of these respective points describe four surfaces for each of which the mean curvature is equal to $1/a$.

II. When the quadric Q is a paraboloid of revolution the loci of the points F_1' and φ_1' are minimal surfaces.

The announcement of these theorems marked the beginning of a new epoch in the theory of the deformation of surfaces. During the following years many eminent geometers attacked problems suggested by them, and in particular the deformation of the general quadric. This problem was solved in the admirable memoirs of Bianchi and Guichard which were jointly crowned by the French Academy in 1908. It is in the exposition of this theory and allied questions that the second edition of Bianchi-Lukat's Vorlesungen differs from the first, and is "vermehrt und verbessert." For the sake of brevity we shall refer to the two editions as (I) and (II).

Chapters 21 and 22 of (I) which deal respectively with N-dimensional space of constant curvature and hypersurfaces in spaces of constant curvature are omitted from (II), in order to make room for the theory of deformation of quadrics. It is to be regretted that this abridgment was necessary and that the treatise did not appear in two volumes so that this important theory, as well as topics subsequently referred to, could have been included. In view of this omission practically all of the quadratic differential forms which appear in the book involve only two variables and so there is no necessity of developing the theory for n variables as in (I). One finds accordingly that in (II) Chapter 2 is given over to the theory of quadratic differential forms in two variables. This makes the treatment simpler in places, but we cannot see that there was any gain made by the change. There is introduced an algebraic proof of the theorem: Two binary quadratic differential forms with the same constant curvature are always equivalent, and the transformation equations involve three arbitrary constants.

The other changes in the first part of the book are a few additions to the theory of deformation in general. Those double families of lines upon any surface S which become asymptotic lines on one of the deforms of S are called *virtual asymptotic lines*. The determination of these families of curves requires the solution of two simultaneous partial differential equations of the second order. Applying the theory of the Picard method of successive approximations,

it is shown that any two intersecting analytic curves C and C' on an analytic surface S determine uniquely a double family of virtual asymptotic lines on S to which these curves belong, one to each family, and the deform of S on which these curves are asymptotic lines is analytic. This theory is applied in more detail to pseudospherical surfaces in Chapter 17, but it is similar to the treatment of this particular case given in Chapter 17 of (I).

At the end of Chapter 8 one finds a theorem due to Chieffi which is of great service in the subsequent discussion. If S is a non-ruled surface applicable to a ruled surface R, to the straight lines on R correspond a family of geodesics on S, called g. Let a be an asymptotic line of either system on S and draw the tangents to the curves g at points a, thus forming a ruled surface R_1. The theorem of Chieffi is that R_1 is applicable to S in such a way that a remains rigid during the deformation.

In Chapter 17 we begin to find traces of the results growing out of the theorems of Guichard. After developing the theory of Bäcklund transformations of pseudospherical surfaces as found in (I) (in places the treatment is slightly different), the author considers the real transformations obtained by combining conjugate imaginary ones, and more particularly the result of taking two opposite transformations B_σ and $B_{-\sigma}$ in applying the "theorem of permutability" (Vertauschbarkeitssatz). Let two such transformations be applied to a pseudospherical surface S, and let S_1 and S_2 be the resulting surfaces and S' the fourth surface which is the transform of these two; the normals to S_1 and S_2 meet in a point M_0 equidistant from these surfaces and likewise the normals to S and S' in a point M_0'. The two surfaces S_0 and S_0', the respective loci of M_0 and M_0', are applicable to the same surface of revolution and are complementary to one another. However, S_0 is not a pseudospherical surface as stated on page 483. According as σ is zero, real and different from zero, or pure imaginary, the meridian curve of the surface of revolution is different, when the latter surface is real. But for all three types of σ the surface S_0 is applicable to the surface of revolution of an imaginary ellipse. Thus one is brought incidentally into contact with the problem of deformation of quadrics.

With the exception of the first five pages the material in

Chapter 18 does not appear in (I). By means of a change of variables the equations of the preceding chapter are given a form which define a Bäcklund transformation of spherical surfaces, that is, surfaces applicable to the sphere, with the difference that all of the first transforms are imaginary. However, these may be combined into real transformations as described in the following theorem: If S_1 is an imaginary Bäcklund transform of a spherical surface S by means of a transformation B_{σ_1}, $(\sigma_1 = a + ib)$, and S_2 is the transform of S by $B_{\bar{\sigma}_1}$, $(\bar{\sigma}_1 = a - ib)$, the fourth surface S' given by the "theorem of permutability" is a real spherical surface. When $b = 0$, the normals to S and S' meet in a point M_0, and the normals to S_1 and S_2 in a point M_0'; the surfaces S_0 and S_0', the loci of these points, are complementary surfaces and are applicable to the same prolate ellipsoid of revolution. When $b = \frac{1}{2}\pi$, S_0 and S_0' are applicable to a hyperboloid of revolution of two sheets.

The second part of Chapter 18 is given over to the proof of the two theorems of Guichard stated at the beginning of this review, and converses of these theorems. In preparation for this discussion the author considers congruences of spheres, whose surface of centers S_0 undergoes a deformation, and also congruences of lines invariably fixed to a surface which is being deformed. If one of the sheets of the envelope of spheres has constant mean curvature as S_0 is deformed, the same is true of the other sheet. If two sheets are minimal surfaces, S_0 is applicable to a paraboloid of revolution P, the radius of a sphere being the distance between the corresponding point on P and the focus. If the mean curvature of the two sheets is different from zero and constant, S_0 is applicable to a prolate ellipsoid of revolution E or a two sheeted hyperboloid of revolution H, and the radius of a sphere is the distance between the corresponding point on E or H and a focus.

The Bäcklund transformations of a pseudospherical or a spherical surface S may be looked upon as transforming the surface elements of S, that is the points of S (called the *centers* of the elements) and small portions of the tangent planes about the points, into ∞^3 surface elements in space which can be grouped so as to be the surface elements of ∞^1 surfaces, the transforms of S. This point of view is not peculiar to these transformations. As a matter of fact it is thus that the author sets up the transformations of surfaces applicable

to any quadric in the following very simple and remarkable manner: Let Q be any quadric and Q_1 a confocal quadric; let a surface element f have for center F, a point of Q, and for its plane π the tangent plane to Q at F; this plane π meets Q_1 in a curve C; take a point F_1 of C as the center of the surface element f_1 whose plane π_1 is determined by the line FF_1 and one of the rulings of Q_1 through F_1, it being understood that rulings of the same family are used in setting up the surface element for each point of Q; the plane π_1 is tangent to Q_1 at some point of the ruling. The fundamental theorem is the following:

THEOREM (A): *If Q is deformed in any manner into a surface S and in the deformation to each element f there are invariably coupled ∞^1 elements f_1, obtained by varying the point F_1 of C for each point F, these ∞^3 surface-elements can be correlated into a unique family of ∞^1 surfaces S_1, each of which is applicable to S and to Q.*

This transformation which converts S into a surface S_1 is called a transformation B_k, where the subscript k is a constant determining Q_1 among the quadrics confocal to Q. The proof of this theorem for the case when Q and Q_1 are hyperbolic paraboloids occupies, with certain allied problems, the whole of Chapter 19. The processes are in keeping with the general treatment of problems of differential geometry in the former chapters of the treatise and as a matter of fact it is essentially that followed by Bianchi in his prize memoir. The coordinates of Q and Q_1 are given in explicit form in terms of parameters referring to their generators. By straightforward steps the relations between the surface elements f and f_1 are established and it is shown that the correlation of the ∞^3 elements f_1, arising from the ∞^2 elements determined by a surface S, into ∞^1 surfaces S_1 requires the integration of an equation of the Riccati type. When the linear element of S_1 is calculated, it is found that it is not of the same form as that of S; consequently S and S_1 are not applicable in such a way that the centers of corresponding surface elements correspond in applicability, if at all. Since Q and Q_1 are confocal paraboloids of the same family, the orthogonal trajectories of the family establish a correspondence between Q and Q_1, namely corresponding points lie on the same trajectory. This is the transformation of Ivory of the one quadric into the other; it plays a very important role in this theory,

as we shall see. Let S be any deform of Q and S_1 one of the surfaces defined by the correlation of the displaced surface elements f_1. If M_1 on S_1 corresponds to F_1 on Q_1, and $\overline{M}$ on S to the point F on Q which is the Ivory transform of F_1, the surfaces S and S_1 are applicable with the points $\overline{M}$ and M_1 in correspondence. The proof of this result covers ten pages most of which involves extensive calculation, so that at times the reader's interest lags, but throughout the whole treatment the geometrical ideas are extremely clear. There is no indication of the reason for hitting upon this correspondence of Ivory as the one leading to applicability. Later in the chapter certain properties of the Ivory transformation are derived as a means to the proof of the theorem: If S_1 is the transform of S by a B_k, then S is a transform of S_1 by the same B_k; these properties may have led to the suggestion of the trial of the transformation as a good guess, but the reader is left in the dark. The development of the results throughout the chapter is simplified and made more interesting by associating with the surfaces S the ruled surfaces R, applicable to S, as derived by means of the Chieffi theorem. Among these results one finds this pretty theorem: If the quadric Q rolls upon an applicable ruled surface R, each of the congruences Γ, $\overline{\Gamma}$ generated by the lines of the first and second system on a confocal quadric Q_1 may be arranged in a unique manner into a simple infinity of ruled surfaces R_1, each of which is applicable to Q and to R; moreover, the applicability of R_1 on R is accomplished by a continuous deformation. The surfaces R and R_1 of the foregoing theorem possess the following property in common with the non-ruled deforms of Q:

THEOREM (B): *The lines MM_1 joining corresponding points on a surface S and a surface S_1, resulting from S by a transformation B_k, constitute a W-congruence of which S and S_1 are the focal surfaces.*

The remainder of Chapter 19 is given over to the proof of Theorem (B) for the case where Q is a hyperbolic paraboloid. The method is much simplified by use of the results for the ruled surfaces R and R_1.

In Chapter 20 Theorems (A) and (B) are established for the case where Q and Q_1 are confocal hyperboloids of one sheet. The method followed is similar to that of the preceding chapter with very little abridgment. In fact the

formulas differ only in form, and if anything they are more involved in the present case. Consequently the geometrical results are practically the same in the two chapters. From a historical point of view this fact is very interesting, since the earlier methods which were fruitful in the study of the deforms of paraboloids could not be applied in the case of central quadrics. It should be mentioned, however, that the exceptional case of hyperboloids of revolution leads to a theory of transformation of Bertrand curves analogous to the transformation of Razzaboni.

From the manner of construction of the surface elements f_1 it follows that if Q is real these elements are real only when Q_1 has negative curvature. Hence whatever be the type of Q, if only real, the surfaces S applicable to it admit real transforms S_1. However, the Ivory transformation makes real points on Q_1 correspond to imaginary points on Q, when the latter is not a hyperbolic paraboloid or a hyperboloid of one sheet. In this case the applicability of S and S_1 is *ideal.* These cases are treated in Chapter 21. By a slight change of notation the formulas of the two former chapters are used to establish Theorems (A) and (B) for the elliptic paraboloid, the hyperboloid of two sheets, and the ellipsoid. A novel converse result is that any real surface applicable in the ideal sense to a quadric of any of these three types admits of ∞^1 transforms S_1 applicable to the same quadric in the real sense. In a like manner there are real surfaces applicable in the ideal sense to the hyperbolic paraboloid and the hyperboloid of one sheet; such a surface is transformable by a B_k into a real surface applicable to the same quadric in the ideal sense. The discussion of these transformations closes with the following "theorem of permutability:"

If S is a surface applicable to a quadric Q, and S_1 and S_2 are two transforms of S by means of transformations B_{k_1} and B_{k_2}, where k_1 and k_2 are different constants, there is a fourth surface S', also applicable to Q, which arises from S_1 and S_2 by transformations B_{k_2}' and B_{k_1}', respectively.

For the proof of this theorem the reader is referred to the third volume of the second Italian edition. Here and in the memoir presented to the French Academy one will find also a treatment of surfaces applicable to the imaginary quadrics. One is inclined to feel that owing to the similarity of treatment of the several cases and in view of the existence of these

details elsewhere much of Chapter 20 and part of Chapter 21 should have been omitted, and the space thus gained should have been used for other material appearing in the second Italian edition which sets forth new ideas. For example, there might have been included the treatment of surfaces with plane or spherical lines of curvature, or an exposition of the Weingarten method of deformation. If the latter had been developed in the manner of the second Italian edition, the ideas and formulas of moving axes would have been introduced, although rather late. We feel that the omission of this operator from the German editions is a serious one. It is to be greatly regretted that both the German and Italian second editions were not rewritten to such an extent that the moving axes could have been used to advantage.

The last three chapters deal with triply orthogonal systems of surfaces. The contents and treatment in the first two of these chapters are essentially the same as in the first edition. By far the greater part of the second chapter deals with confocal quadrics and geodesics on quadrics. This material could have been introduced into Chapter 6 and it would have served as a fine illustration of the general theory there set forth. We have always felt that this should have been done in the first editions. In fact we cannot see the reason for concealing this subject matter in a chapter which a beginner is not likely to read. The last chapter of (I) deals with Lamé families of pseudospherical surfaces. In the last chapter of (II) the treatment of spherical surfaces as well is included, together with that of certain surfaces applicable to quadrics.

We were pleased to find a fairly complete index in addition to a table of contents, which appears in (I) alone. The translation is well done, there are very few typographical errors and the bookmaking is up to the standard of the Teubners.

LUTHER PFAHLER EISENHART.

NOTES.

THE April number (volume 13, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "A generalization of Weierstrass's preparation theorem for a power series in several variables," by G. A. BLISS; "An existence theorem for periodic solutions," by W. D. MACMILLAN; "A condition that a function in a projective space be rational," by W. F. OSGOOD; "A method of proving certain theorems relating to rational functions which are adjoint to an algebraic equation for a given value of the independent variable," by J. C. FIELDS: "The dependence of focal points upon curvature for problems of the calculus of variations in space," by M. B. WHITE; "Orthocentric properties of the plane directed n-line," by J. E. HODGSON; "Certain singularities of point transformations in space of three dimensions," by S. E. URNER; "Multiple correspondences determined by the rational plane quintic curve," by J. R. CONNER.

THE April number (volume 34, number 2) of the *American Journal of Mathematics* contains: "The groups of birational transformations of algebraic curves of genus 5," by J. V. MCKELVEY; "The general theory of linear q-difference equations," by R. D. CARMICHAEL; "Concerning linear projective order," by A. R. SCHWEITZER; "On certain expansions of elliptic, hyperelliptic, and related periodic functions," by F. R. MOULTON; "On cubic birational space transformations," by H. P. HUDSON; "On the definition of multiplication of irrational numbers," by OSWALD VEBLEN.

THE Library of Congress will have ready for distribution in July the first of a series of catalogue cards forming the beginning of a dictionary catalogue of all articles in the Encyklopädie der mathematischen Wissenschaften and the Encyclopédie des Sciences mathématiques. The material for all parts of the two Encyclopedias thus far issued has already been supplied by Brown University, which will continue to supply the material as future parts appear.

AT the meeting of the London mathematical society held on March 14 the following paper was read: By G. T. BENNETT, "The cubic surface as a degenerate quartic."

THE date of the Fifth international congress of mathematicians at Cambridge, England, is August 22–28. Series of lectures are announced by E. BOREL, E. W. BROWN, A. KNESER, E. G. H. LANDAU, J. LARMOR, and Sir W. WHITE. The American members of the organizing committee are Professors M. Bôcher, J. C. Fields, E. H. Moore, F. R. Moulton, B. O. Peirce, J. Pierpont, and A. G. Webster. Requests for information regarding the congress should be addressed to the general secretary of the organizing committee, Professor E. W. Hobson, Christ's College, Cambridge.

THE academy of sciences of Paris announces that the grand prize (3000 frs.) will be awarded in 1914 for the solution of the following problem. "To perfect the theory of functions of one variable which can be represented by trigonometric series of several arguments which are linear functions of the variable. The academy will be pleased to consider important applications to mathematical physics and to celestial mechanics." The following problem is proposed for the Fourneyron prize (1000 frs.) in 1914. "Theoretical and experimental study of turbines."

THE royal academy of sciences of Belgium proposes the following prize problems for 1913: "An important contribution to the infinitesimal geometry of curved surfaces" (800 frs.). "Summarize the theory of systems of space cubic curves and make a new contribution to the theory" (800 frs.). Competing memoirs should be sent to the secretary before August 1, 1913.

THE first congress of Russian mathematicians, including university professors and teachers in the secondary schools was held at St. Petersburg, January 6–16, 1912, in which more than 1200 persons participated. The president of the congress, Professor A. VASSILIEF, delivered an able address "On instruction in mathematics and philosophy in the secondary schools." Extracts of the papers will appear in an early number of *Enseignement Mathématique*.

THE following university courses in mathematics are offered during the summer semester of 1912.

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ:

Space curves and surfaces, four hours; Analytic functions, four hours; Hypergeometric series, two hours; Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Determinants, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Quadric surfaces, four hours; Riemann's theory, four hours; Seminar, two hours. — By Professor R. LEHMANN-FILHÉS: Mechanics, four hours; with exercises, one hour. — By Professor G. HETTNER: Introduction to ordinary differential equations, two hours. — By Professor J. KNOBLAUCH: Applications of elliptic functions, four hours; Integral calculus, four hours; Selected chapters of analytic geometry, one hour. — By Professor I. SCHUR: Elliptic functions, four hours; Differential calculus, four hours; with exercises, one hour.

UNIVERSITY OF BONN. — By Professor E. STUDY: Differential and integral calculus, four hours; with exercises, one hour; Quaternions, two hours; Seminar, two hours. — By Professor F. LONDON: Descriptive geometry, four hours; Analytic geometry, II, three hours; with exercises, one hour. — By Dr. J. O. MÜLLER: Theory of numbers, three hours.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Analytic geometry, four hours; Seminar, two hours.—By Professor D. HILBERT: Ordinary differential equations, four hours; Mathematical foundation of physics, four hours.—By Professor C. RUNGE: Mechanics, II, six hours; Seminar, two hours.—By Professor E. LANDAU: Algebraic numbers and ideals, four hours; The Galois theory of equations, two hours; Mathematical games and recreations, one hour; Seminar, two hours.—By Professor F. BERNSTEIN: Theory of probabilities, three hours; Mathematics of insurance, two hours; Seminar, two hours.—By Dr. O. TOEPLITZ: Differential and integral calculus, with exercises, six hours; Theory of an infinite number of variables, two hours.—By Dr. A. HAAR: Celestial mechanics, two hours.—By Dr. E. WEYL: Curves and surfaces, four hours; Theory of real variables, two hours; Exercises in analytic geometry, two hours.—By Dr. C. v. KÁRMÁN: Applications of the calculus of variations, three hours.—By Dr. R. SCHIMMACH: The international commission on mathematical instruction, two hours.

THE following university courses in mathematics are announced for the academic year 1912–1913.

COLUMBIA UNIVERSITY. — By Professor C. J. KEYSER: Modern theories in geometry, three hours; History and significance of central mathematical concepts, three hours. — By Professor T. S. FISKE: Introduction to the theory of functions of a real variable, three hours; Functions defined by linear differential equations, three hours. — By Professor F. N. COLE: Introduction to the theory of functions, three hours; Theory of plane curves, three hours. — By Professor JAMES MACLAY: Theory of numbers, three hours, first half-year; Differential equations, three hours, second half-year. — By Professor D. E. SMITH: History of mathematics, three hours. — By Professor EDWARD KASNER: Integral equations, two hours; Seminar in differential geometry, three hours. — By Professor W. B. FITE: Calculus of variations, three hours. — By Professor H. E. HAWKES: Modern higher algebra, three hours, second half-year. — By Dr. H. W. REDDICK: Differential equations, three hours, first half-year. — By Dr. N. J. LENNES: Projective geometry, three hours.

The mathematical colloquium will meet at intervals of about two weeks.

CORNELL UNIVERSITY.—By Professor J. MCMAHON: Theory of probabilities, two hours; Vector analysis, two hours.—By Professor J. I. HUTCHINSON: Linear differential equations, two hours; Elliptic integrals, two hours (first term).—By Professor V. SNYDER: Analytic geometry of space, three hours (second term).—By Professor F. R. SHARPE: Introduction to mathematical physics, three hours.—By Professor W. B. CARVER: Theory of equations, two hours; Descriptive geometry, three hours (first term).—By Professor A. RANUM: Differential geometry, two hours.—By Professor D. C. GILLESPIE: Principles of mechanics, three hours.—By Dr. C. F. CRAIG: Elementary differential equations, two hours.—By Dr. F. W. OWENS: Projective geometry, three hours.—By Dr. J. V. MCKELVEY: Algebraic curves, three hours.—By Dr. L. L. SILVERMAN: Advanced calculus, three hours.—By Dr. W. A. HURWITZ: Theory of functions of a complex variable, three hours.

THE following doctorates in mathematics were conferred by the French universities during the academic year 1910–1911:

Paris, Doctorat d'Etat.

CAUBET, P. "Etude des principales inégalités du mouvement de la lune qui dépendent de l'inclinaison."

CHAZY, J. "Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes."

SIRE, J. "Sur les fonctions entières de deux variables d'ordre apparent total fini."

CHATELET, A. "Sur certains ensembles de tableaux et leur application à la théorie des nombres."

GAU, E. "Sur l'intégration des équations aux dérivées partielles du second ordre par la méthode de M. Darboux."

ANNYCKE, TH. "Contribution à l'étude thermomechanique des tiges et des plaques."

GARNIER, R. "Sur les équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes."

JANISZEWSKI, S. "Sur les continus irréductibles entre deux points."

VILLAT, H. "Sur la résistance des fluides."

Doctorat d'Université.

GRAMONT, A. de. "Essai d'aérodynamique du plan."

LEMAIRE, P. "Théorie des compas gyroscopiques."

Nancy, Doctorat d'Etat.

ARNOULT, J. "Sur le mouvement d'un fil dans l'espace."

Lille, Doctorat d'Etat.

BARRÉ, E. "Sur une classe de solutions des équations indéfinies de l'équilibre d'élasticité," et "Application de la géométrie cinématique à la théorie des surfaces engendrées par une courbe variable."

PROFESSOR G. B. GUCCIA, of the University of Palermo, has been elected a national member of the Royal Institute of Venice.

PROFESSOR G. DARBOUX, of the University of Paris, has been elected an honorary member of the Royal Irish academy.

DR. L. N. G. FILON has been appointed professor of applied mathematics at the University of London.

THE mathematical prize of the Prince Jablonowski society for 1911 has been awarded to Professor ERNST NEUMANN.

PROFESSOR W. KUTTA, of the technical high school at Aachen, has been appointed full professor of mathematics at the technical high school at Stuttgart.

PROFESSOR G. D. BIRKHOFF, of Princeton University, has accepted a call to an assistant professorship of mathematics at Harvard University.

AT the University of Missouri Dr. LOUIS INGOLD has been promoted to an assistant professorship of mathematics.

AT Columbia University Dr. L. P. SICELOFF has been promoted to an assistant professorship of mathematics. Mr. C. A. FISCHER has been appointed instructor in mathematics.

MR. R. W. BURGESS has been appointed assistant in mathematics at Cornell University.

AT Dartmouth College DR. E. G. BILL, of Purdue University, has been appointed assistant professor of mathematics. Drs. J. E. ROWE, of Haverford College, and F. M. MORGAN, of Cornell University, have been appointed instructors in mathematics.

PROFESSOR J. AMSLER-LAFFON, inventor of the polar planimeter, died at Schaffhausen January 3, at the age of 89 years.

DR. EMIL LEMOINE, recipient of the Francoeur prize for several years, died at Paris, February 21, at the age of 72 years.

PROFESSOR OSBORNE REYNOLDS, of Owens College (now the University of Manchester) died February 21, at the age of 70 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BERRY (W. J.). Differential equations of the first species. New York, Van Nostrand, 1912. 12mo. Cloth.

BOSMANS (H.). Notes sur l'arithmétique de Simon Stevin. Louvain, Ceuterick, 1911.

BROLL (G.). Mehrdeutige Verwandtschaften zwischen Punktfeldern, insbesondere solche, welche durch Raumkurven veranlasst werden. (Diss.) Breslau, 1911. 8vo. 86 pp.

BÜCHER, Neue, über Naturwissenschaften und Mathematik. Die Neuigkeiten des deutschen Buchhandels, nach Wissenschaften geordnet. Mitgeteilt Winter 1911–1912. Leipzig, Hinrichs, 1912. 8vo. pp. 69–92. M. 0.30

CAJORI (F.). Notes on the history of geometry and algebra. Boston, Heath, 1911. 8vo. 25 pp. Paper. $0.10

CZUBER (E.). Vorlesungen über Differential- und Integral rechnung. 1ter Band. 3te, sorgfältig durchgesehene Auflage. Leipzig, Teubner, 1912. 8vo. 14+605 pp. Cloth. M. 12.00

ENGELHARDT (P.). Ueber die graphische und näherungsweise numerische Auflösung von Gleichungen 2ten und 3ten Grades mit einer Unbekannten im Unterrichte an der Mittelschule. (Progr.) Würzburg, 1911. 8vo. 33 pp.

FOERSTER (O.). Ueber Cassinische Kurven auf der Pseudosphäre. (Diss.) Münster, 1911.

GRANDJEAN (K.). Ueber die mit einer Schläflischen Doppelsechs zusammenhängende Fläche F^3. (Diss.) Strassburg, 1911.

HEIBERG (J. L.). Naturwissenschaften und Mathematik im klassischen Altertum. (Aus Natur und Geisteswelt. Nr. 370.) Leipzig, Teubner, 1912. 8vo. 4+102 pp. Cloth. M. 1.25

JOACHIMI (O.). Ueber Kurven bei denen die beiden Krümmungen durch eine quadratische Beziehung verknüpft sind. (Diss.) Münster, 1911.

KLEBER (A.). Ueber einige mehrdeutige Verwandtschaften zweier Ebenen. (Diss.) Rostock, 1911.

KLEEBERG (R.). Ueber die Diskriminantenflächen der Gleichungen $A\cos x + B\sin 2x + C\cos 2x + D\cos 3x = 0$. (Diss.) München, 1911. 8vo. 87 pp.

KRAFT (K.). Das Normalenproblem an Kurven und Flächen zweiter Ordnung in den endlichen Raumformen. (Diss.) Münster, 1911.

LÖWENHERZ (A.). Die Frenet'schen Formeln im R_{n+1}. (Diss.) Königsberg, 1911. 8vo. 73 pp.

MOHRMANN (H.). Normalflächen und projektive Gruppen. (Habilitationsschrift.) Karlsruhe, 1911. 4to. 30 pp.

MÜHE (A.). Beweis des Fermatschen Prinzips. Berlin, Reimer, 1911. 8vo. 3 pp. M. 0.30

OSTER (E.). Zentralperspektive, stereographische Projektion und quadratische Binärformen. (Diss.) Strassburg, 1911.

PERL (E.). Untersuchungen über Differenzenkoeffizienten erster und zweiter Art, insbesondere über ihren Zusammenhang mit verwandten Grössen. (Diss.) Königsberg, 1911. 8vo. 126 pp.

RATSCHLÄGE für die Studierenden der Mathematik und Physik an der Universität Jena. 3te Auflage. Jena, 1912.

ROGERS (R. A. P.). See SALMON (G.).

SALMON (G.). A treatise on the analytic geometry of three dimensions. Revised by R. A. P. Rogers. 5th edition. In two volumes. Volume 1. New York, Longmans, 1912. 8vo. 22+470 pp. $3.00

SCHOENFLIES (A.). Ueber die allgemeinen Beziehungen der Mathematik zur Kultur und Erkenntnis. (Antrittsrede.) Frankfurt, Auffarth, 1912.

STELZNER (O.). Mathematik und Naturwissenschaften an den neuhumanistischen Schulen, unter Einwirkung von Gesner, Ernesti, Heine und Wolf. Ein Beitrag zur Geschichte des mathematisch-naturwissenschaftlichen Unterrichts. (Diss.) Leipzig, 1911.

STOFFAËS (Abbé E.). Cours de mathématiques supérieures. 3e édition entièrement refondue. Tome 2: Courbes et surfaces. Equations différentielles. Paris, Gauthier-Villars, 1911. 8vo. 365 pp. Fr. 10.00

STURM (A.). Geschichte der Mathematik bis zum Ausgange des 18ten Jahrhunderts. (Sammlung Göschen, Nr. 226.) 2te, verbesserte Auflage. Leipzig, Göschen, 1911. 8vo. 155 pp. Cloth. M. 0.80

THUE (A.). Einige Bemerkungen über die Gleichung $Ax^3 + By^3 = Cz^3$. (Videnskapsselskapets skrifter.) Utgift for Fridtjof Nansens fond. Kristiania, Dybwad, 1911. 8vo. 7 pp. M. 0.50

——. Ueber einige in ganzen Zahlen x und y unmögliche Gleichungen $F(x, y) = 0$. Utgift for Fridtjof Nansens fond. Kristiania, Dybwad, 1911. 8vo. 7 pp. M. 0.50

TOSCHI DI FAGNANO (G. C. De'). Opere matematiche, pubblicate sotto gli auspicî della società italiana per il progresso delle scienze dai soci V. Volterra, G. Loria, D. Gambioli. Volumi I–II, che contengono la materia dei tomi I e II delle Produzioni matematiche. Milano, Albrighi, 1911. 8vo. 9+474 pp; 9+471 pp.

UMFAHRER (J.). Beweis der Richtigkeit des "Grossen Fermatschen Satzes." München, Eher, 1912. 8vo. 16 pp. M. 0.50

II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

BARNARD (S.) and CHILD (J. M.). A new algebra. Volume 2, containing parts 4–6. With answers. London, Macmillan, 1912. 8vo. 4s.

BELIKOW (A.) und NATHING (A.). Lehrbuch der Algebra. 2ter Teil, bearbeitet von A. Nathing. 3te Auflage. St. Petersburg, Eggers, 1911. 8vo. 3+179 pp. M. 3.00

BERICHTE über den mathematischen Unterricht in Österreich. 10tes Heft, von O. Simony, E. Kobald, A. Mikuta, und K. Reich. Wien, Hölder, 1911. 8vo. 39 pp. M. 1.20

BOS (H.). Géométrie élémentaire. 22e édition. Paris, Hachette, 1912. 16 mo. 288 pp. Fr. 2.00

CHILD (J. M.). See BARNARD (S.).

CONVERSE (H. A.). See DURELL (F.).

DURELL (F.). Plane and spherical trigonometry. With chapters on plane surveying by H. A. Converse. Also logarithmic and trigonometric tables. New York, Merrill, 1912. 8vo. 295+32 pp. $1.40

——. School algebra. New York, Merrill, 1911. 12mo. 507 pp. $1.10

ENRIQUES (F.) e AMALDI (U.). Elementi di geometria, 4a edizione ridotta. Bologna, Zanichelli, 1911. 16mo. 13+351 pp. L. 2.50

——. Elementi di geometria, ad uso delle scuole secondarie superiori. 5a edizione. Bologna, Zanichelli, 1911. 16mo. 11+623 pp. L. 4.50

FERGUSSON (J. C.). The percentage unit of angular measurement, with logarithms. New York, Longmans, 1912. 67+666 pp. $20.00

GEBHARDT. Der mathematische und physikalische Unterricht auf dem Vitzthumschen Gymnasium zu Dresden während des Zeitraumes 1861 bis 1911. Leipzig, Teubner, 1911.

GEOMETRIA, La, nel catasto italiano. (VI congresso nazionale fra geometri, agronomi, e periti agrimensori italiani.) Pesaro, Terenzi, 1911. 8vo. 10 pp.

GIUDICE (M. del). Lezioni di aritmetica razionale e algebra elementare. Volume I: Fondamenti della dottrina del numero naturale e frazionario. Roma, Marcolli, 1911. 8vo. 23+184 pp. L. 3.50

GRASSI (F.). See SERRET (G. A.).

HANNYGTON (J. C.). Table of logarithms and anti-logarithms. Four figures. London, Layton, 1912. 8vo. 46 pp. Limp. 1s. 6d.

JONES (S. I.). Mathematical wrinkles. A book of knotty problems, mathematical recreations, rules of mensuration, etc. Gunter, Texas, 1912. 12mo. Half leather. $1.50

KOSCHEMANN und OTTEN. Lehr- und Uebungsbuch für den mathematischen Unterricht an Mittelschulen. 2ter Teil: Arithmetik und Algebra. 3ter Teil: Geometrie. Frankfurt, Diesterweg, 1911. M. 4.10

KURRAS (K.). Maturitäts-Aufgaben aus der Mathematik nebst ausführlichen Lösungen. Berlin, Ulrich, 1912. 8vo. 72 pp. M. 2.00

LÖRCHER (O.). Methodisches Lehr- und Uebungsbuch der Algebra. Stuttgart, Grub, 1912. 8vo. 7+142 pp. Cloth. M. 2.00

MARKS (C. I.). See MATHEMATICAL questions.

MATHEMATICAL questions and solutions from the Educational Times. Edited by C. I. Marks. New series, volume 20. London, Hodgson, 1912. 8vo. 6s. 6d.

NARDI (P.). Geometria. Parte II: Stereometria. 2a edizione, riveduta. Livorno, Giusti, 1912. 16mo. 72 pp. L. 0.70

NATHING (A.). Lehrbuch der Stereometrie. 2te, unveränderte Auflage. St. Petersburg, Eggers, 1912. 8vo. 3+91 pp. M. 3.00

——. See Belikow (A.).

PINCHERLE (S.). Lezioni di algebra elementare. Bologna, Zanichelli, 1911. 8vo. 389 pp. L. 4.50

RIBONI (G.). Elementi di geometria. 8a edizione, con modificazioni ed aggiunte. Bologna, Zanichelli, 1911. 16mo. 7+366 pp. L. 2.50

SCHMEHL (C.). Lehrbuch der Stereometrie für höhere Lehranstalten. Giessen, Roth, 1912. M. 2.00

SCHMITT (P.). Die Behandlung der Raumlehre in der Volksschule. 3te Auflage. Langensalza, Schulbuchhandlung, 1911. 8vo. 4+151 pp. M. 1.75

SCHULTZE (A.). The teaching of mathematics. New York, Macmillan, 1912. 12mo. Cloth.

SCOTT (E. E.). Tables of logarithms and anti-logarithms to five places. London, Layton, 1912. 8vo. 383 pp. 5s.

SERRET (G. A.). Elementi di trigonometria. Per cura di G. Tolomei. 12a edizione. Firenze, Le Monnier, 1911. 8vo. 150 pp. L. 1.75

——. Trattato di trigonometria. Traduzione, con modificazioni ed aggiunte, da G. Tolomei. Firenze, Le Monnier, 1911. 8vo. 269 pp. L. 2.25

——. Trattato di trigonometria piana e sferica, tradotto da F. Grassi. 7a edizione. Torino, Gallizio, 1911. 8vo. 8+312 pp. L. 3.00

SMITH (D. E.). See SUZZALLO (H.).

SOCCI (A.) e TOLOMEI (G.). Elementi di geometria, secondo il metodo di Euclide. Volume I. 18a edizione. Firenze, Le Monnier, 1912. 8vo. 21+160 pp. L. 1.75

——. Elementi di matematica. Volume I. 6a edizione, completamente rifatta. Firenze, Le Monnier, 1912. 8vo. 4+204 pp. L. 1.50

——. Elementi di matematica. Volume III. 5a impressione. Firenze, Le Monnier, 1911. 8vo. 206 pp. L. 1.50

——. Nozioni intuitive di geometria elementare. 8a impressione. Firenze, Le Monnier, 1912. 8vo. 112 pp. L. 1.00

SUZZALLO (H.). The teaching of primary arithmetic . A critical study of recent tendencies in method. With an introduction by D. E. Smith. (Riverside educational monographs.) Boston, Houghton Mifflin, 1911. 16mo. 10+123 pp. $0.60

TOLOMEI (G.). See SERRET (G. A.) and SOCCI (A.).

TREUTLEIN (P.). See WIENER (H.).

WELLS (W.). Complete trigonometry. Revised edition. Boston, Heath, 1911. 12mo. 6+163+9+23 pp. $1.08

WIENER (H.). und TREUTLEIN (P.). Verzeichnis Mathematischer Modelle für Hochschulen. 2te Ausgabe. Leipzig, Teubner, 1912.

III. APPLIED MATHEMATICS.

ABRAHAM (M.). Theorie der Elektrizität. 2 Bände. 1ter Band: Einführung in die Maxwellsche Theorie der Elektrizität. Mit einem einleitenden Abschnitte von A. Föppl. 4te Auflage. Leipzig, Teubner, 1912. 8vo. 18+410 pp. M. 12.00

ALKINDI, TIDEUS und PSEUDO-EUKLID. Drei optische Werke. Herausgegeben und erklärt von A. A. Björnbo und S. Vogl. Mit einem Gedächtniswort auf A. A. Björnbo von G. H. Zeuthen, einem Ver-

zeichnis seiner Schriften und seinem Bildnis. (Abhandlungen zur Geschichte der mathematischen Wissenschaften. Heft 26, Nr. 3.) Leipzig, Teubner, 1912. 8vo. 176 pp. M. 10.00

BJÖRNBO (A. A.). See ALKINDI.

BRAUER (E. A.). See EULER (L.).

BROWN (Sir H.). Irrigation. Its principles and practice as a branch of engineering. 2d edition, revised. London, Constable, 1912. 8vo. 318 pp. 16s.

CAPPILLERI (A.). See KLAUSER.

COHN (E.). Physikalisches über Raum und Zeit. (Vortrag.) Leipzig, Teubner, 1911. 8vo. 24 pp. M. 0.60

DARLING (C. R.). Heat for engineers. A treatise on heat. 2d edition revised. London, Spon, 1912. 8vo. 475 pp. 12s. 6d.

DEUSS (P.). Untersuchungen der Dupré-Rankine'schen Dampfspannungsformel an dem neuen Material von S. Young und mit Rücksicht auf die Theorie der korrespondierenden Zustände. (Diss.). Zürich, 1910. 8vo. 55 pp.

DUHEM (P.). Die Wandlungen der Mechanik und der mechanischen Naturerklärung. Uebersetzt von P. Frank unter Mitwirkung von E. Stiasny. Leipzig, 1912. M. 6.40.

EULER (L.). Vollständigere Theorie der Maschinen, die durch Reaktion des Wassers in Bewegung versetzt werden. Herausgegeben von E. A. Brauer und M. Winkelmann. Leipzig, 1911. M. 1.60

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FLEMING (J. A.). Waves and ripples in water, air, and æther. 2d issue, revised. London, S.P.C.K., 1912. 8vo. 312 pp. 2s. 6d.

FÖPPL (A.). See ABRAHAM (M.).

FRANK (P.). See DUHEM (P.).

GEISSLER (J.). Die Gleichgewichtsbedingungen der Raummechanik mit besonderer Berucksichtigung der elektrischen, magnetischen und Gravitations-erscheinungen. (Diss.) Rostock, 1911. 8vo. 86 pp.

GRAETZ (L.). Die Elektrizität und ihre Anwendungen. 16te Auflage. Stuttgart, Engelhorn, 1912. 8vo. 16+722 pp. Cloth. M. 9.00

HAMEL (G.). Elementare Mechanik. Leipzig, Teubner, 1912. 8vo. 18+634 pp. Cloth. M. 18.00

JAMIESON (A.). A textbook of applied mechanics. Volume 4: Hydraulics. 8th edition, thoroughly revised and enlarged. London, Griffin, 1912. 8vo. 342 pp. 5s.

KIRSCHKE (A.). Die darstellende Geometrie des Maschinen-technikers. Leipzig, 1912. M. 1.20

KLAUSER und LAHN. Lehrbuch der Vermessungskunde, bearbeitet von A. Cappilleri. 4te Auflage. Wien, Deuticke, 1912. Kr. 4.00

KRÜGER (L.). Konforme Abbildung des Erdellipsoids in der Ebene. (Veröffentlichung des Königlich Preussischen Geodätischen Institutes. Neue Folge. Nr. 52.) Leipzig, Teubner, 1912. M. 9.00

LAMLA (E.). Ueber die Hydrodynamik des Relativitäts-prinzips. (Diss.) Berlin, 1912.

LINEHAM (W. J.). A textbook of mechanical engineering. 11th edition, revised. London, Chapman and Hall, 1912. 8vo. 1268 pp. 12s. 6d.

LORENZ (H.). Neue Theorie und Berechnung der Kreiselräder, Wasser- und Dampfturbinen, u. s. w. 2te, neu bearbeitete und vermehrte Auflage. München, Oldembourg, 1911. M. 11.00

MARCOLONGO (R.). Theoretische Mechanik. Deutsch von H. E. Timerding. 2ter Band: Dynamik und Mechanik der deformierbaren Körper. Leipzig, Teubner, 1912. 8vo. 7+344 pp. Cloth. M. 11.00

MILLER (M. W.). Descriptive geometry. Champaign, Ill., 1911. 12mo. 127+20 pp. $1.50

NEUMANN (F.). Gesammelte Werke. Herausgegeben von seinen Schülern. 3ter Band. Leipzig, Teubner, 1912. M. 36.00

NICHOLSON (E.). Men and measures. A history of weights and measures, ancient and modern. London, Smith and Elder, 1912. 8vo. 12+313 pp. 7s. 6d.

NUTTING (P. G.). Outlines of applied optics. Philadelphia, Blakiston, 1912. 12mo. 9+234 pp. $2.00

PLANCK (M.). Ueber neuere thermodynamische Theorien (Nernstsches Wärmetheorem und Quantenhypothese). (Vortrag.) Leipzig, Allgemeine Verlagsgesellschaft, 1912. 8vo. 34 pp. M. 1.60.

SATTERLY (J.). See STEWART (R. W.).

SCHNEIDER (H.). Die Energie der aus glühendem CaO entweichenden Elektronen. (Diss.) Berlin, 1911. 8vo. 40 pp.

SCHÖNHÖFER (R.). Statische Untersuchung von Bogen- und Wölbtragwerken nach den Grundsätzen der Elastizitätstheorie. 2te, neu bearbeitete und erweiterte Auflage. Berlin, Ernst, 1911. M. 2.60

STEUER (K.). Untersuchungen von Dampfdiagrammen auf Grund der Dynamik der Dampfströmung in der Kolbendampfmaschine. (Diss.) Danzig, 1911. 8vo. 43 pp.

STEWART (R. W.) and SATTERLY (J.). Senior sound and light. London, Clive, 1912. 8vo. 480 pp. 4s.

STOCKMANN (W.). Untersuchung der von Condensatorfunken ausgehenden gedämpften Schallwellen. (Diss.) Breslau, 1911. 8vo. 47 pp.

TIMERDING (H. E.). See MARCOLONGO (R.).

VILLAMIL (R. de). A, B, C of hydrodynamics. London, Spon, 1912. 8vo. 148 pp.

WINKELMANN (M.). See EULER (L.).

ZEUTHEN (G. H.). See ALKINDI.

ZIWET (A.) and FIELD (P.). Introduction to analytical mechanics. New York, Macmillan, 1912. 12mo. 376 pp. Cloth. $1.60